

STRUCTURED, COMPACTLY SUPPORTED BANACH FRAME DECOMPOSITIONS OF DECOMPOSITION SPACES

FELIX VOIGTLAENDER

ABSTRACT. We present a very general framework for the construction of structured, possibly compactly supported Banach frames and atomic decompositions for a given decomposition space. Here, a decomposition space $\mathcal{D}(\mathcal{Q}, L^p, \ell_w^q)$ is defined essentially like a classical Besov space, but the usual dyadic covering is replaced by an (almost) arbitrary covering $\mathcal{Q} = (Q_i)_{i \in I}$. Thus, if $\Phi = (\varphi_i)_{i \in I}$ is a suitable partition of unity subordinate to \mathcal{Q} , then $\|g\|_{\mathcal{D}(\mathcal{Q}, L^p, \ell_w^q)} := \left\| \left(\|\mathcal{F}^{-1}(\varphi_i \hat{g})\|_{L^p} \right)_{i \in I} \right\|_{\ell_w^q}$. Special cases include the class of Besov spaces and (α) -modulation spaces, as well as a large class of wavelet-type coorbit spaces and so-called shearlet smoothness spaces.

Assuming that the covering \mathcal{Q} is of the regular form $\mathcal{Q} = (T_i Q + b_i)_{i \in I}$, with $T_i \in \text{GL}(\mathbb{R}^d)$, $b_i \in \mathbb{R}^d$, we fix a *prototype function* $\gamma \in L^1(\mathbb{R}^d)$ and consider the structured generalized shift invariant system

$$\Psi_\delta := \left(L_{\delta \cdot T_i^{-1} T_k} \gamma^{[i]} \right)_{i \in I, k \in \mathbb{Z}^d} \quad \text{with} \quad \gamma^{[i]} := |\det T_i|^{1/2} \cdot M_{b_i}(\gamma \circ T_i^T),$$

where L_x and M_ξ denote translation and modulation, respectively. The main contribution of the paper is to provide verifiable conditions on the prototype γ which ensure that Ψ_δ forms, respectively, a Banach frame or an atomic decomposition for the space $\mathcal{D}(\mathcal{Q}, L^p, \ell_w^q)$, for sufficiently small *sampling density* $\delta > 0$. Crucially, while the decomposition space $\mathcal{D}(\mathcal{Q}, L^p, \ell_w^q)$ is defined using the *bandlimited* family Φ , the construction presented here usually allows for the prototype γ to be *compactly supported* in space. We emphasize that the theory presented here can cover the whole range $p, q \in (0, \infty]$ and not only the case $p, q \in [1, \infty]$ of Banach spaces.

An important feature of our theory is that in many cases, the system Ψ_δ will *simultaneously* form a Banach frame and an atomic decomposition for $\mathcal{D}(\mathcal{Q}, L^p, \ell_w^q)$. This implies that for frames of the form Ψ_δ , *analysis sparsity is equivalent to synthesis sparsity*, i.e., the analysis coefficients $\left(\left\langle f, L_{\delta \cdot T_i^{-1} T_k} \gamma^{[i]} \right\rangle \right)_{i \in I, k \in \mathbb{Z}^d}$ lie in $\ell^p(I \times \mathbb{Z}^d)$ if and only if f is an element of a certain decomposition space, if and only if $f = \sum_{i \in I, k \in \mathbb{Z}^d} \left[c_k^{(i)} \cdot L_{\delta \cdot T_i^{-1} T_k} \gamma^{[i]} \right]$ for some sequence $(c_k^{(i)})_{i \in I, k \in \mathbb{Z}^d} \in \ell^p(I \times \mathbb{Z}^d)$. This is very convenient, since for many frame constructions—like shearlets—one only knows that the *analysis coefficients* for a class of “nice” signals are sparse. This, however, only entails synthesis sparsity with respect to the *dual* frame, about which often only limited knowledge is available. Using the theory presented here, one can derive synthesis sparsity with respect to the *primal* frame, for which one has an explicit formula and whose properties like smoothness and time-frequency localization are well understood.

As a sample application, we show that the developed theory applies to α -modulation spaces and to (inhomogeneous) Besov spaces. In a companion paper, we also show that the theory applies to shearlet smoothness spaces.

1. INTRODUCTION

In this section, we first motivate and describe our approach for the construction of structured Banach frame decompositions for decomposition spaces and compare our results to the known literature. Then, we introduce a few standard and non-standard conventions and standing assumptions which are used in the remainder of the paper. Finally, we give a brief overview over the structure of the paper.

1.1. Motivation and comparison to known results. Given a Banach space X , a family $\Psi = (\psi_i)_{i \in I}$ in X' is called a **Banach frame**[44] for X if there is a **solid sequence space** $Y \leq \mathbb{C}^I$ such that

- the **analysis operator** $A_\Psi : X \rightarrow Y, x \mapsto \left(\langle x, \psi_i \rangle_{X, X'} \right)_{i \in I}$ is well-defined and bounded,
- there is a bounded linear **reconstruction operator** $R : Y \rightarrow X$ satisfying $R \circ A_\Psi = \text{id}_X$.

In particular, this implies $\|x\|_X \asymp \|A_\Psi x\|_Y$ uniformly over $x \in X$. Here, a Banach space $Y \leq \mathbb{C}^I$ is called **solid** if for all sequences $x = (x_i)_{i \in I}$ and $y = (y_i)_{i \in I}$ with $y \in Y$ and $|x_i| \leq |y_i|$ for all $i \in I$, it follows that $x \in Y$ with $\|x\|_Y \leq \|y\|_Y$.

Dual to the notion of a Banach frame, a family $\Phi = (\varphi_i)_{i \in I}$ in X is called an **atomic decomposition**[44] for X if there is a solid sequence space $Z \leq \mathbb{C}^I$ such that

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- the **synthesis operator** $S_\Phi : Z \rightarrow X, (x_i)_{i \in I} \mapsto \sum_{i \in I} x_i \varphi_i$ is well-defined and bounded, where convergence of the series occurs in a suitable (weak) sense,
- there is a bounded linear **coefficient operator** $C : X \rightarrow Z$ satisfying $S_\Phi \circ C = \text{id}_X$.

In particular, this implies that every $x \in X$ can be written as $x = \sum_{i \in I} c_i \varphi_i$ for a suitable sequence $c = (c_i)_{i \in I} = Cx$.

The existence of nice Banach frames and atomic decompositions for a given (family of) Banach space(s) is extremely convenient, since the study of many properties like existence of embeddings, boundedness of operators and description of interpolation spaces, etc., of the Banach spaces under consideration can be reduced to studying these properties for the associated sequence spaces, which are often much easier to understand.

For this reason, much effort has been spent to derive existence of Banach frames and atomic decompositions for many well-known spaces like Besov- and Sobolev spaces. The most well-known types of (Banach) frames are probably the **wavelet characterization** of Besov spaces (see e.g. [74, Theorem 1.64]), the closely related characterization of these spaces using the φ -transform [34, 33], as well as the existence of **Gabor frames** for modulation spaces[45].

1.1.1. Classical group-based coorbit theory. By generalizing the similarities between the theories of wavelet- and Gabor frames, Feichtinger and Gröchenig initiated the study of so-called **coorbit spaces**[25, 26, 27, 44], which provide a systematic way of obtaining Banach frames and atomic decompositions for certain Banach spaces. Precisely, one starts with an irreducible, (square)-integrable representation $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ of some locally compact Hausdorff (LCH) topological group G . This representation induces for each $g \in \mathcal{H}$ an associated **voice transform**

$$V_g : \mathcal{H} \rightarrow C(G), f \mapsto V_g f \quad \text{where} \quad (V_g f)(x) = \langle f, \pi(x)g \rangle_{\mathcal{H}}.$$

For an **admissible vector** $\psi \in \mathcal{H} \setminus \{0\}$ (which means $V_\psi \psi \in L^2(G)$), it follows[20] that $V_\psi : \mathcal{H} \rightarrow L^2(G)$ is (a scalar multiple of) an isometry, so that $(\pi(x)\psi)_{x \in G}$ is a **tight continuous frame** for \mathcal{H} , since

$$\|f\|_{\mathcal{H}}^2 = C_\psi \cdot \int_G |(V_\psi f)(x)|^2 d\mu(x) \quad \forall f \in \mathcal{H}.$$

In particular, this identity implies $\mathcal{H} = \{f \mid V_\psi f \in L^2(G)\}$. In generalization of this identity, coorbit theory shows that for “good enough” *analyzing windows* ψ and each suitable, *solid* function space $Y \leq L_{\text{loc}}^1(G)$, one can define the associated **coorbit space** as

$$\text{Co}(Y) := \{f \in \mathcal{R} \mid V_\psi f \in Y\} \quad \text{with norm} \quad \|f\|_{\text{Co}(Y)} = \|V_\psi f\|_Y.$$

Here, $\mathcal{R} = \mathcal{R}_Y$ is a suitable *reservoir*. Informally, \mathcal{R} corresponds to the set of (tempered) distributions; but due to the generality in which coorbit spaces are defined, one has to use a slightly different definition, intrinsic to the group G , cf. [26, Section 4].

The main statement of coorbit theory is that one can *discretize* the (continuous, tight) frame $(\pi(x)\psi)_{x \in G}$, to obtain discrete Banach frames and atomic decompositions, simultaneously for all spaces $\text{Co}(Y)$, where Y ranges over a suitable set of solid function spaces on Y . More precisely, the following are true:

- For each translation invariant, solid function space $Y \leq L_{\text{loc}}^1(G)$, there is a so-called **control weight** $w = w_Y : G \rightarrow (0, \infty)$, cf. [26, equation (4.10)]. For the following statements, we always assume that w is a control weight for Y .
- Associated to each control weight w , there is a class $\mathcal{B}_w \subset \mathcal{H}$ of **good (analyzing) vectors** such that for each two $\psi_1, \psi_2 \in \mathcal{B}_w \setminus \{0\}$, the identity

$$\{f \in \mathcal{R} \mid V_{\psi_1} f \in Y\} = \text{Co}(Y) = \{f \in \mathcal{R} \mid V_{\psi_2} f \in Y\}$$

holds, i.e., one has a *consistency statement*.

- For each control weight w and each $\psi \in \mathcal{B}_w \setminus \{0\}$, there is a unit neighborhood $U = U(\psi, w) \subset G$, such that for every U -dense and **relatively separated** family $X = (x_i)_{i \in I}$ in G , the family $(\pi(x_i)\psi)_{i \in I}$ forms an **atomic decomposition** of $\text{Co}(Y)$, i.e., there is a solid, discrete sequence space $Y_d(X) \leq \mathbb{C}^I$ associated to Y such that the following hold (see [26, Theorem 6.1 and the associated remark]):
 - the synthesis operator

$$S : Y_d(X) \rightarrow \text{Co}(Y), (\lambda_i)_{i \in I} \mapsto \sum_{i \in I} \lambda_i \cdot \pi(x_i) \psi$$

is well-defined and bounded (with convergence in the weak- $*$ -topology of the reservoir \mathcal{R}),

- there is a bounded linear operator $C : \text{Co}(Y) \rightarrow Y_d(X)$ satisfying $S \circ C = \text{id}_{\text{Co}(Y)}$.

- For each control weight w and each $\psi \in \mathcal{B}_w \setminus \{0\}$, there is a unit neighborhood $V = V(\psi, w) \subset G$ such that for every V -dense and relatively separated family $X = (x_i)_{i \in I}$ in G , the family $(\pi(x_i)\psi)_{i \in I}$ forms a **Banach frame** for $\text{Co}(Y)$, i.e., with the same solid sequence space $Y_d(X)$ as above, the following hold (see [44, Theorem 5.3]):
 - the analysis operator

$$A : \text{Co}(Y) \rightarrow Y_d(X), f \mapsto (\langle f, \pi(x_i)\psi \rangle)_{i \in I}$$

is well-defined and bounded,

- there is a bounded linear operator $R : Y_d(X) \rightarrow \text{Co}(Y)$ satisfying $R \circ A = \text{id}_{\text{Co}(Y)}$.

Here, a family $X = (x_i)_{i \in I}$ is called **V -dense** if $G = \bigcup_{i \in I} x_i V$ and **relatively separated** if it is a finite union of separated sets, where a family $Z = (z_j)_{j \in J}$ is called **separated** if there is a unit neighborhood $W \subset G$ satisfying $z_j W \cap z_\ell W = \emptyset$ for $j \neq \ell$.

Among other examples, this *group-based* coorbit theory can be used to obtain Banach frames and atomic decompositions for modulation spaces as well as for *homogeneous* Besov spaces. There are also several extensions, for example to the setting of Quasi-Banach spaces[67], and to the setting of possibly reducible or non-integrable group representations[10].

The main limitation of this theory, however, is that many relevant spaces like *inhomogeneous* Besov spaces are *not* covered by it.

1.1.2. Generalized coorbit theory. To overcome this limitation, Fornasier, Rauhut and Ullrich[31, 69] developed what is called **generalized coorbit theory**; see also [4] for some corrections and extensions and [53] for a generalization to Quasi-Banach spaces. For generalized coorbit theory, one starts from a Hilbert space \mathcal{H} , for which one is given a **continuous frame** $\Psi = (\psi_x)_{x \in X}$ which is indexed by some locally compact measure space X , equipped with a Radon measure μ . Formally, this means that for each $f \in \mathcal{H}$, the function $X \rightarrow \mathbb{C}, x \mapsto \langle f, \psi_x \rangle_{\mathcal{H}}$ is measurable and there are constants $0 < A \leq B$ satisfying

$$A \cdot \|f\|_{\mathcal{H}}^2 \leq \int_X |\langle f, \psi_x \rangle_{\mathcal{H}}|^2 d\mu(x) \leq B \cdot \|f\|_{\mathcal{H}}^2 \quad \forall f \in \mathcal{H}.$$

In this case, the **frame operator** $S : \mathcal{H} \rightarrow \mathcal{H}, f \mapsto \int_X \langle f, \psi_x \rangle_{\mathcal{H}} \cdot \psi_x d\mu(x)$ (with the integral understood in the weak sense) is well-defined, self-adjoint and positive and thus invertible. Hence, one can form the **canonical dual frame** $\tilde{\Psi} = (\tilde{\psi}_x)_{x \in X} = (S^{-1}\psi_x)_{x \in X}$. If the frame Ψ is **tight**, one can choose $A = B$ and the dual frame $\tilde{\Psi}$ is simply a scalar multiple of Ψ , but in general, Ψ and $\tilde{\Psi}$ might be very different. For generalized coorbit theory to be applicable at all, one requires the **cross-gramian kernel**

$$R : X \times X \rightarrow \mathbb{C}, (x, y) \mapsto \langle \psi_y, S^{-1}\psi_x \rangle = \langle \psi_y, \tilde{\psi}_x \rangle$$

to have certain decay/mapping properties; precisely, one requires $R \in \mathcal{A}_m$, where $m = m_Y$ is a given control weight associated to the solid function space $Y \leq L_{\text{loc}}^1(X)$ in which one is interested. Here, \mathcal{A}_m is a suitable algebra of kernels, cf. [31, Section 3].

With these two frames $\Psi, \tilde{\Psi}$, there are *two* associated voice transforms, given by

$$V_{\Psi}f(x) := \langle f, \psi_x \rangle_{\mathcal{H}} \quad \text{and} \quad W_{\Psi}f(x) := \langle f, \tilde{\psi}_x \rangle_{\mathcal{H}} = (V_{\Psi}[S^{-1}f])(x),$$

and then (cf. [31, equation (3.8) and Definition 3.1]) also *two* reservoirs $\mathcal{R}_1 := (\mathcal{K}_v^1)^{\top}$ and $\mathcal{R}_2 := (\mathcal{H}_v^1)^{\top}$ and *two* coorbit spaces

$$\text{Co}(Y) := \{f \in \mathcal{R}_1 \mid V_{\Psi}f \in Y\} \quad \text{and} \quad \widetilde{\text{Co}}(Y) := \{f \in \mathcal{R}_2 \mid W_{\Psi}f \in Y\}.$$

Then, if the frame Ψ is “good enough” (the precise meaning of which depends on the space Y), one can again discretize the continuous frame Ψ to obtain atomic decompositions and Banach frames. However, one has to be a bit careful; assuming that the family $(x_i)_{i \in I}$ is “dense enough in X ” (cf. [31, Theorem 5.7] for the details), we have the following:

- the family $(\psi_{x_i})_{i \in I}$ is an atomic decomposition of $\widetilde{\text{Co}}(Y)$ with corresponding sequence space Y^{\flat} ,
- the family $(\psi_{x_i})_{i \in I}$ is a Banach frame for $\text{Co}(Y)$ with corresponding sequence space Y^{\flat} .

Thus, although generalized coorbit theory is immensely powerful and general, its main limitation is that one essentially has to start from a *tight* continuous frame for a Hilbert space \mathcal{H} , since in the non-tight case one faces several limitations:

- In most cases of continuous *non-tight* frames, one knows very little about the properties of the (canonical) dual frame $\tilde{\Psi}$, which makes it hard to verify that the kernel $R(x, y) = \langle \psi_y, \tilde{\psi}_x \rangle$ satisfies $R \in \mathcal{A}_m$.

- As seen above, one is faced with two *distinct* coorbit spaces $\text{Co}(Y)$ and $\widetilde{\text{Co}}(Y)$ and obtains a Banach frame for $\text{Co}(Y)$ and an atomic decomposition for $\widetilde{\text{Co}}(Y)$. In many cases, however, it is desired to *simultaneously* have a Banach frame and an atomic decomposition for *one* common space.

We mention that [31, Section 4] provides criteria which ensure $\text{Co}(Y) = \widetilde{\text{Co}}(Y)$, namely if Ψ and $\tilde{\Psi}$ are \mathcal{A}_m -self-localized. To show that this is true, however, one again needs to know a lot about the dual frame $\tilde{\Psi}$, which in general one does not. The most convenient way out (outlined in [31, Theorem 4.7 and the comments afterward]) is to find a suitable **spectral** subalgebra \mathcal{A} of \mathcal{A}_m and then to show that the kernel $K(x, y) = \langle \psi_y, \psi_x \rangle$ satisfies $K \in \mathcal{A}$. Once this is shown, [31, Theorem 4.7] yields $\text{Co}(Y) = \widetilde{\text{Co}}(Y)$ as well as $R \in \mathcal{A} \subset \mathcal{A}_m$, so that coorbit theory is applicable. The main limitation of this approach is that not too many spectral algebras of kernels are known.

In total, there are two desirable use cases of (generalized) coorbit theory in which an actual application is *difficult*, or even *impossible*:

- (1) In the first case, one is given a (family of) Banach space(s) B and wants to find Banach frames and atomic decompositions for B . To achieve this via (generalized) coorbit theory, one has to find a (preferably tight) *continuous* frame $\Psi = (\psi_x)_{x \in X}$ and a (family of) solid Banach function space(s) $Y \leq L^1_{\text{loc}}(X)$ such that $B = \text{Co}(Y) = \{f \mid V_\Psi f \in Y\}$. Furthermore, one has to verify that Ψ indeed satisfies all prerequisites for the application of generalized coorbit theory. Finally, if Ψ is non-tight, one has to verify $\text{Co}(Y) = \widetilde{\text{Co}}(Y)$, for example by using the approach using spectral algebras which we outlined above.
- (2) In the second case, which occurs e.g. if one wants to study the approximation theoretic properties of discrete, cone-adapted shearlet frames[54], one starts with a *discrete* frame $\Psi_d = (\psi_i)_{i \in I}$ (or with a family of such discrete frames, e.g., parametrized by the sampling density) for a Hilbert space \mathcal{H} and one wants to understand the space of those functions which are *analysis-sparse* with respect to this frame, e.g., the space

$$B_q := \{f \in \mathcal{H} \mid (\langle f, \psi_i \rangle)_{i \in I} \in \ell^q(I)\} \quad \text{for} \quad q < 2.$$

An important property one might be interested in is *whether analysis sparsity is equivalent to synthesis sparsity*, i.e., whether every $f \in B_q$ admits an expansion $f = \sum_{i \in I} c_i \psi_i$ for a sequence $c = (c_i)_{i \in I} \in \ell^q(I)$.

To derive such a statement using coorbit theory, one needs to find a continuous (preferably tight) frame $\Psi = (\psi_x)_{x \in X}$ for \mathcal{H} such that the discretization $(\psi_{x_i})_{i \in I}$ of this frame (in the sense of generalized coorbit theory) is equal to Ψ_d . Then, provided that $\text{Co}(Y) = \widetilde{\text{Co}}(Y) = B_q$, coorbit theory will yield the desired statement.

The main problem here—as witnessed by the example of discrete cone-adapted shearlets—is that it can often be very hard or even impossible to find such a continuous frame Ψ , much less a tight one. There are tight continuous shearlet frames, e.g. those related to shearlet coorbit spaces[14, 18, 17, 12], but a discretization of these frames does *not* yield discrete *cone-adapted* shearlet systems.

As we will see now, our approach does *not* require to have a continuous frame which can then be discretized. Thus, in this aspect, our approach improves upon (generalized) coorbit theory. As we will see in the companion paper [66], we are in particular able to handle discrete cone-adapted shearlet frames; and for this case, our theory indeed shows that *analysis sparsity is equivalent to synthesis sparsity*.

1.1.3. Our approach using decomposition spaces. For our approach, we start with a **structured covering** \mathcal{Q} of (an open subset \mathcal{O} of) the frequency space \mathbb{R}^d . More precisely (see Subsection 1.3 for the completely formal assumptions), we assume that

$$\mathcal{Q} = (Q_i)_{i \in I} = (T_i Q + b_i)_{i \in I} \tag{1.1}$$

for a fixed open, precompact set $Q \subset \mathbb{R}^d$ and certain linear maps $T_i \in \text{GL}(\mathbb{R}^d)$ and translations $b_i \in \mathbb{R}^d$. Then, given a suitable partition of unity $\Phi = (\varphi_i)_{i \in I}$ subordinate to \mathcal{Q} and a suitable weight $w = (w_i)_{i \in I}$ on I , as well as $p, q \in (0, \infty]$, one defines the decomposition space (quasi)-norm of a distribution g as

$$\|g\|_{\mathcal{D}(\mathcal{Q}, L^p, \ell_w^q)} := \left\| \left(\|\mathcal{F}^{-1}(\varphi_i \cdot \widehat{g})\|_{L^p} \right)_{i \in I} \right\|_{\ell_w^q} = \left\| \left(\|(\mathcal{F}^{-1} \varphi_i) * g\|_{L^p} \right)_{i \in I} \right\|_{\ell_w^q},$$

while the **decomposition space** $\mathcal{D}(\mathcal{Q}, L^p, \ell_w^q)$ consists of all distributions for which this (quasi)-norm is finite. For the exact interpretation of “distribution” in this context, we refer to Subsection 1.3. In words, the decomposition space norm is computed by first decomposing g in frequency according to the covering \mathcal{Q} to obtain the pieces $g_i = \mathcal{F}^{-1}(\varphi_i \cdot \widehat{g})$. Each of these pieces is then measured in L^p and the overall norm is a certain ℓ_w^q -norm over all of these contributions. In most of the paper, we will even consider the weighted L^p -spaces L^p_v instead of L^p . But in this introduction, we will mostly stick to the setting just described, for the sake of simplicity.

Our general aim is to show that one can obtain compactly supported Banach frames and atomic decompositions Ψ of a very special, structured form for the decomposition space $\mathcal{D}(\mathcal{Q}, L^p, \ell_w^q)$. In fact, it will turn out that the system Ψ can be taken to be a generalized shift invariant system generated by a *single prototype function* γ , similar to the way in which a prototype function can generate Gabor, wavelet and shearlet systems.

To see exactly how such a system Ψ might look like, let us write $S_i \xi := T_i \xi + b_i$ for $i \in I$. Note that if $\text{supp } \hat{\gamma} \subset Q$, then $\text{supp } [\hat{\gamma} \circ S_i^{-1}] \subset Q_i$. The same remains true in a weak sense if the strict inclusion $\text{supp } \hat{\gamma} \subset Q$ is replaced by requiring that $\hat{\gamma}$ be *essentially* supported in Q , which can even hold if γ is not band-limited. Now, note $\mathcal{F}\gamma^{(i)} = \hat{\gamma} \circ S_i^{-1}$ for $\gamma^{(i)} := |\det T_i| \cdot M_{b_i} [\gamma \circ T_i^T]$. For consistency with the L^2 -setting, we also consider

$$\gamma^{[i]} := |\det T_i|^{1/2} \cdot M_{b_i} [\gamma \circ T_i^T]. \quad (1.2)$$

In fact, we will even allow the generator γ to vary with $i \in I$, i.e., $\gamma^{[i]} = |\det T_i|^{1/2} \cdot M_{b_i} [\gamma_i \circ T_i^T]$. An example where this is useful is an inhomogeneous wavelet system: If the generator γ is required to be *independent* of $i \in I$, the “low-pass part” of the wavelet system needs to be obtained by a frequency shift (i.e., by a modulation) from the mother wavelet γ . Indeed, since we consider only affine dilations of $\hat{\gamma}$ and since any *linear* dilation of $\hat{\gamma}$ will vanish at the origin, this is the only way in which one can cover the origin of the frequency domain. In most cases, the exact shape of the low-pass part is not important, so that taking a modulation of the mother wavelet is acceptable. But in other cases, one might desire more specific properties of the low-pass part; for example, one could want it to be real-valued. In this case, the added flexibility of allowing γ to depend on $i \in I$ might be valuable. In this introduction, however, we will only consider the case in which $\gamma_i = \gamma$ is independent of $i \in I$, for the sake of simplicity.

Now, since the family $(\widehat{\gamma^{(i)}})_{i \in I}$ behaves similarly to the family $(\varphi_i)_{i \in I}$ (at least with respect to the (essential) frequency support), one could be tempted to conjecture that

$$\|g\|_{\mathcal{D}(\mathcal{Q}, L^p, \ell_w^q)} \asymp \|(\|\gamma^{(i)} * g\|_{L^p})_{i \in I}\|_{\ell_w^q}. \quad (1.3)$$

For the special case of α -modulation spaces, this statement was established (for (almost) arbitrary $\gamma \in \mathcal{S}(\mathbb{R}^d)$) in [52]. Our first result (cf. Section 3) will be to show that for $p \in [1, \infty]$, equation (1.3) is indeed valid under suitable assumptions on γ . Furthermore, for $p \in (0, 1)$, we have the slightly modified statement

$$\|g\|_{\mathcal{D}(\mathcal{Q}, L^p, \ell_w^q)} \asymp \left\| \left(\|\gamma^{(i)} * g\|_{W_{T_i^{-T}[-1,1]^d}(L^p)} \right)_{i \in I} \right\|_{\ell_w^q}, \quad (1.4)$$

where $W_{T_i^{-T}[-1,1]^d}(L^p)$ is a so-called **Wiener amalgam space** (originally introduced by Feichtinger[22]). We refer to the results (1.3)-(1.4) as stating that the family $(\gamma^{(i)})_{i \in I}$ forms a **semi-discrete Banach frame** for $\mathcal{D}(\mathcal{Q}, L^p, \ell_w^q)$. The reason for this nomenclature is that the index set of the family $((\gamma^{(i)} * g)(x))_{i \in I, x \in \mathbb{R}^d}$ has the discrete part I , but also the continuous part \mathbb{R}^d .

Our next results are concerned with a further discretization of this semi-discrete Banach frame. Indeed, under more stringent assumptions on γ , we show in Section 4 for $\delta > 0$ sufficiently small that the **structured generalized shift-invariant system**

$$\Psi_\delta := \left(L_{\delta \cdot T_i^{-T} k} \widetilde{\gamma^{[i]}} \right)_{i \in I, k \in \mathbb{Z}^d} \quad \text{with} \quad \tilde{f}(x) := f(-x) \quad (1.5)$$

generates a Banach frame for $\mathcal{D}(\mathcal{Q}, L^p, \ell_w^q)$, with the associated discrete sequence space

$$Y := \ell^q \left(|\det T_i|^{\frac{1}{2} - \frac{1}{p}} \cdot w_i \right)_{i \in I} \left([L^p(\mathbb{Z}^d)]_{i \in I} \right) \quad \text{where} \quad \left\| (c_k^{(i)})_{i \in I, k \in \mathbb{Z}^d} \right\|_Y = \left\| \left(|\det T_i|^{\frac{1}{2} - \frac{1}{p}} \cdot w_i \cdot \|(c_k^{(i)})_{k \in \mathbb{Z}^d}\|_{\ell^p} \right)_{i \in I} \right\|_{\ell^q}. \quad (1.6)$$

Since the system Ψ_δ is generated in a very structured way—similar to the usual definition of Gabor, wavelet or shearlet frames—from a *single* prototype function, we call Ψ_δ a **structured Banach frame** for $\mathcal{D}(\mathcal{Q}, L^p, \ell_w^q)$.

Finally, we show in Section 5—again under slightly different assumptions on γ —that the family $(L_{\delta \cdot T_i^{-T} k} \gamma^{[i]})_{i \in I, k \in \mathbb{Z}^d}$ forms an **atomic decomposition** for $\mathcal{D}(\mathcal{Q}, L^p, \ell_w^q)$, with the same associated sequence space Y as above. As above, we call this family a **structured atomic decomposition**. Hence, at least if γ is symmetric and fulfills certain technical conditions, the family Ψ_δ will *simultaneously* form a Banach frame, as well as an atomic decomposition for $\mathcal{D}(\mathcal{Q}, L^p, \ell_w^q)$; in particular, this implies that *analysis sparsity is equivalent to synthesis sparsity* for Ψ_δ .

We remark that the assumptions placed on the prototype function γ are quite technical, even though we will achieve a significant simplification of these conditions in Section 6. Indeed, a slightly simplified version of our main theorem concerning Banach frames reads as follows:

Theorem. (cf. Corollary 6.6 for the precise statement)

Recall from equation (1.1) that $\mathcal{Q} = (T_i Q + b_i)_{i \in I}$. Assume that there is an open set $P \subset \mathbb{R}^d$ with $\overline{P} \subset Q$ and $\mathcal{O} = \bigcup_{i \in I} T_i P + b_i$. Let $p, q \in (0, \infty]$. Then there are explicitly given $N \in \mathbb{N}$ and $\sigma, \tau > 0$, depending on d, p, q , with the following property:

If $w = (w_i)_{i \in I}$ is a \mathcal{Q} -moderate¹ weight and if $\gamma \in L^1(\mathbb{R}^d)$ satisfies the following:

- (1) We have $\widehat{\gamma} \in C^\infty(\mathbb{R}^d)$, where all partial derivatives of $\widehat{\gamma}$ are polynomially bounded.
- (2) We have $\gamma \in C^1(\mathbb{R}^d)$ and $\partial_\ell \gamma \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ for all $\ell \in \{1, \dots, d\}$.
- (3) We have $\widehat{\gamma}(\xi) \neq 0$ for all $\xi \in \overline{\mathcal{Q}}$.
- (4) We have

$$C_1 := \sup_{i \in I} \sum_{j \in I} M_{j,i} < \infty \quad \text{and} \quad C_2 := \sup_{j \in I} \sum_{i \in I} M_{j,i} < \infty \quad (1.7)$$

with

$$M_{j,i} := \left(\frac{w_j}{w_i} \right)^\tau \cdot (1 + \|T_j^{-1} T_i\|)^\sigma \cdot \max_{|\beta| \leq 1} \left(|\det T_i|^{-1} \cdot \int_{Q_i} \max_{|\alpha| \leq N} \left| \left(\partial^\alpha \widehat{\partial^\beta \gamma} \right) (T_j^{-1}(\xi - b_j)) \right| d\xi \right)^\tau.$$

Then, for $\delta > 0$ sufficiently small, the family Ψ_δ from equation (1.5) is a Banach frame for $\mathcal{D}(\mathcal{Q}, L^p, \ell_w^q)$, with associated sequence space Y as given in equation (1.6).

The conditions which ensure that γ generates an atomic decomposition are similar, but slightly more complicated, cf. Corollary 6.7. For the sake of brevity, we omit them in this introduction.

Of course, condition (1.7) is quite technical. The main reason for this is that hugely different coverings \mathcal{Q} are treated using the same theory. Thus, given a specific covering (e.g. the ones used to define Besov spaces or α -modulation spaces), the difficulty consists in reducing the general, abstract criteria provided by the theory to readily verifiable criteria involving only the smoothness, decay and Fourier decay of γ . As we will see in Sections 7 and 8, this is indeed possible for Besov spaces and α -modulation spaces. In addition, in the companion paper [66] we will show that the theory also applies to shearlet smoothness spaces. Furthermore, it turns out that in each of these cases one can find *compactly supported* prototype functions γ which fulfill the relevant criteria. Thus, although the decomposition spaces $\mathcal{D}(\mathcal{Q}, L^p, \ell_w^q)$ are defined using the *bandlimited* partition of unity $(\varphi_i)_{i \in I}$, it is usually possible to give alternative characterizations in terms of *compactly supported* functions.

At a first glance, the difficulty pertaining to the technical conditions described above seems be a major drawback of the theory presented here in comparison to coorbit theory. But in fact, coorbit theory faces the same problem: In essentially every example where coorbit theory is applicable, one has a systematic way of assigning to each “prototype” ψ a whole family $\Psi = (\psi_x)_{x \in X}$. Then, one has to obtain a profound understanding of the mapping $\psi \mapsto (\psi_x)_{x \in X}$ in order to derive readily verifiable conditions on ψ which ensure that the family Ψ is suitable for the application of coorbit theory, in particular to ensure that Ψ is a (Hilbert space) frame and that the kernel $R(x, y) = \langle \psi_y, S^{-1} \psi_x \rangle$ belongs to \mathcal{A}_m . As examples for the effort one still has to put in to apply coorbit theory in specific situations, we mention [37, 38, 36, 39, 69, 14, 18, 17, 13, 40, 35, 75]. We emphasize that this effort should not be seen as a shortcoming of the mentioned papers or of (generalized) coorbit theory, but rather as showing that despite the tremendous simplifications coorbit theory has to offer, one still has to put in work to apply it in concrete situations. The same is true of the results in this paper.

In fact, there is an intimate connection between the decomposition space setting considered here and the coorbit setting considered in [14, 17, 38, 39, 75]: In all of these papers, the authors consider coorbit spaces of a semi-direct product $\mathbb{R}^d \rtimes H$ for suitable **dilation groups** $H \leq \text{GL}(\mathbb{R}^d)$, where the associated unitary representation $\pi : \mathbb{R}^d \rtimes H \rightarrow \mathcal{U}(L^2(\mathbb{R}^d))$, $(x, h) \mapsto L_x D_h$ is the **quasi-regular representation**, i.e., the natural action of $\mathbb{R}^d \rtimes H$ on $L^2(\mathbb{R}^d)$ in terms of the translations L_x and the dilations D_h with $D_h f = |\det h|^{-1/2} \cdot (f \circ h^{-1})$. The mentioned papers contain—typically somewhat technical and lengthy—sufficient criteria which ensure that a given *mother wavelet* can serve as an atom in the coorbit scheme. These conditions heavily depend on the considered dilation group H and also on the weight $w : \mathbb{R}^d \rtimes H \rightarrow (0, \infty)$ which is used for the weighted mixed Lebesgue space $L_w^{p,q}(\mathbb{R}^d \rtimes H)$. For a given mother wavelet g satisfying these criteria, the theory of coorbit spaces implies that each sufficiently densely sampled family $(\pi(x_j, h_j)g)_{j \in J}$ yields an atomic decomposition, as well as a Banach frame for the coorbit space $\text{Co}(L_w^{p,q}(\mathbb{R}^d \rtimes H))$.

But as shown in [40] and in [76, Section 4], we have $\text{Co}(L_w^{p,q}(\mathbb{R}^d \rtimes H)) = \mathcal{D}(\mathcal{Q}_H, L^p, \ell_w^q)$ up to canonical identifications, at least if the weight $w = w(x, h)$ only depends on the second factor, i.e., if $w = w(h)$. Here, the

¹cf. Section 1.3, equation (1.13) for the precise definition.

so-called **induced covering** $\mathcal{Q}_H = (h_i^{-T}Q)_{i \in I}$ of the **dual orbit** $\mathcal{O} = H^T \xi_0 \subset \mathbb{R}^d$ is determined by an arbitrary well-spread family $(h_i)_{i \in I}$ in H . Given this identification, one can then apply the theory developed in this paper to derive conditions on the prototype γ which ensure that the family

$$\left(L_{\delta \cdot h_i k} \gamma^{[i]} \right)_{i \in I, k \in \mathbb{Z}^d} = \left(|\det h_i|^{-1/2} \cdot L_{\delta \cdot h_i k} [\gamma \circ h_i^{-1}] \right)_{i \in I, k \in \mathbb{Z}^d} = (\pi(\delta h_i k, h_i) \gamma)_{i \in I, k \in \mathbb{Z}^d}$$

forms a Banach frame, or an atomic decomposition for the decomposition space $\mathcal{D}(\mathcal{Q}_H, L^p, \ell_w^q)$ and thus also for the coorbit space $\text{Co}(L_w^{p,q}(\mathbb{R}^d \rtimes H))$. Note that the family $[(\delta h_i k, h_i)]_{i \in I, k \in \mathbb{Z}^d}$ is well-spread in $\mathbb{R}^d \rtimes H$ since $(h_i)_{i \in I}$ is well-spread in H . Hence, the theory developed in this paper yields Banach frames and atomic decompositions which are *of the same form* as those obtained via coorbit theory. As future work, we plan a systematic comparison of the conditions imposed on the prototype γ by coorbit theory (as in [14, 17, 38, 39, 75]) on the one hand and by the theory developed in this paper on the other hand.

In spite of the strong connection between coorbit theory and the theory developed in this paper, they differ in some important aspects:

As a first difference, we observe that coorbit theory requires to pass from the given continuous frame $(\psi_x)_{x \in X}$ to a sufficiently densely sampled version $(\psi_{x_i})_{i \in I}$. This will usually not only require a sufficiently dense sampling *in the space domain* (which corresponds to δ in equation (1.5)), but also to a rather dense sampling *in the frequency domain*. In contrast, for our approach only the sampling density *in space* needs to be sufficiently high. The “frequency sampling density” is fixed a priori by choosing the covering $\mathcal{Q} = (T_i Q + b_i)_{i \in I}$.

Next, the main advantage of our approach in comparison to coorbit theory is that one does *not* need to start from a given *continuous* frame $(\psi_x)_{x \in X}$ which is then discretized. In fact, one can even start from a given discrete frame which is of the form (1.5). As long as the family $\mathcal{Q} = (T_i Q + b_i)_{i \in I}$ forms a suitable covering, one can then consider the associated decomposition spaces $\mathcal{D}(\mathcal{Q}, L^p, \ell_w^q)$ and use the theory presented here to justify that the discrete frame one started with forms a Banach frame and an atomic decomposition for $\mathcal{D}(\mathcal{Q}, L^p, \ell_w^q)$, possibly after adjusting the sampling density.

Probably, this intuition is what originally lead Labate et al. to the introduction of the **shearlet smoothness spaces**[58], although they did not have the machinery to rigorously prove that the usual discrete, cone-adapted shearlet systems indeed yield Banach frames and atomic decompositions for the shearlet smoothness spaces. Using the theory developed here, we will see in the companion paper [66] that this is indeed the case. Furthermore, we will employ our results to show that suitable discrete, cone-adapted shearlet systems achieve an almost optimal approximation rate for the class of **cartoon-like functions**. At a first glance, this might appear to be a well-known statement, but a closer inspection of the classical results about approximation of cartoon-like functions by shearlets (see e.g. [56, 55, 57, 47]) reveals that these papers in fact only show that the N -term approximation f_N *with respect to the dual frame* of the shearlet frame satisfies the (almost optimal) rate $\|f - f_N\|_{L^2} \lesssim N^{-1} \cdot (\log N)^\theta$ for suitable $\theta > 0$.

1.1.4. Comparison to other constructions of Banach frame decompositions of decomposition spaces. One of the first general constructions of atomic decompositions for decomposition spaces was given by Borup and Nielsen in [9]. The main difference between their approach and ours is that our frame elements $\gamma^{[i]}$ can be chosen to be compactly supported, while Borup and Nielsen purely focus on bandlimited frame elements.

There is also a more recent paper by Nielsen and Rasmussen [63] in which they construct compactly supported frames for certain decomposition spaces. In comparison to that paper, our assumptions concerning the covering \mathcal{Q} are more general, while our conclusions are more specific:

- In [63], the authors only consider coverings \mathcal{Q} which are induced by considering \mathbb{R}^d in a certain way as a space of homogeneous type: More precisely, \mathcal{Q} is assumed to satisfy $\mathcal{Q} = (Q_k)_{k \in \mathbb{Z}^d} = (\mathcal{B}_A(\xi_k, \varrho \cdot h(\xi_k)))_{k \in \mathbb{Z}^d}$, where the balls $\mathcal{B}_A(\xi, r) = \{\zeta \in \mathbb{R}^d \mid |\zeta - \xi|_A < r\}$ are defined using the quasi-metric $|\cdot|_A$, which is induced in a certain way (cf. [63, Definition 2.1]) by the one-parameter group of dilations $(\delta_t)_{t>0}$ where $\delta_t = \exp(A \cdot \ln t)$ for a fixed matrix $A \in \mathbb{R}^{d \times d}$ with positive eigenvalues. As shown between [63, Lemma 2.6] and [63, Definition 2.7], we have

$$Q_k = \delta_{h(\xi_k)} [\mathcal{B}_A(0, \varrho)] + \xi_k \quad \forall k \in \mathbb{Z}^d,$$

so that all sets Q_k of the covering \mathcal{Q} are affine images of a fixed set, where the linear parts of the affine maps are all elements of the one-parameter family $(\delta_t)_{t>0}$. Note with $\nu := \text{trace } A > 0$ that $\det \delta_t = t^\nu$ for all $t > 0$, so that δ_t is uniquely determined by its determinant. Since the covering used to define shearlet smoothness spaces uses affine transformations for which many *different* linear parts have the *same* determinant, this shows—or at least very strongly indicates—that the covering used to define the shearlet smoothness spaces does *not* satisfy the assumptions imposed in [63], while our theory is able to handle these spaces.

Below, we will give another more rigorous argument which shows that the theory developed in [63] does in fact *neither* apply to the usual dyadic covering which is used to define (inhomogeneous) Besov spaces, *nor* to the covering used to define shearlet smoothness spaces.

- While each of the compactly supported Banach frames constructed in [63] is a union of generalized shift invariant systems, it is *not* true that the frames are generated from a single prototype function in the same structured way as in our paper. In contrast, the Banach frames constructed in [63] are of the form

$$(\psi_{k,n})_{k,n \in \mathbb{Z}^d} = \left([h(\xi_k)]^{\nu/2} \cdot \tau_k \left(\delta_{h(\xi_k)}^T \bullet - \frac{\pi}{a} n \right) e^{i\langle \cdot, \xi_k \rangle} \right)_{k,n \in \mathbb{Z}^d} \quad \text{where} \quad \tau_k = \sum_{i=1}^K a_i^{(k)} g_m(\bullet + b_i^{(k)}),$$

for suitable $K, m \in \mathbb{N}$ and with $g_m = C_g m^\nu \cdot g \circ \delta_m^T$. Hence, using notation as in eq. (1.2) with the covering \mathcal{Q} as defined above and with $b_k := \xi_k$, as well as $T_k := \delta_{h(\xi_k)}$, we have $(\psi_{k,n})_{k,n \in \mathbb{Z}^d} = \left(L_{\frac{\pi}{a} T_k^{-T} n} \tau_k^{[k]} \right)_{k,n \in \mathbb{Z}^d}$, while the structured family Ψ_δ defined in equation (1.5) satisfies $\Psi_\delta = \left(L_{\delta \cdot T_k^{-T} n} \widehat{\psi^{[k]}} \right)_{k,n \in \mathbb{Z}^d}$. In other words, while the structured Banach frames constructed in this paper arise from a *single* prototype function by translations, modulations and dilations, the frames constructed in [63] do *not* satisfy this property.

In particular, if the covering \mathcal{Q} is the usual dyadic covering of \mathbb{R}^d used to define (inhomogeneous) Besov spaces, then Ψ_δ will be an (inhomogeneous) wavelet frame, while this is *not* in general true of the frame constructed in [63]. Additionally, the results in [63] are not applicable in this setting, as we will see below.

This last defect—that the resulting Banach frame is not generated from a *single* prototype—is addressed in the follow-up paper [62]. There, Morten Nielsen considers the same general setting as described above. He then constructs a *bandlimited* Banach frame for the associated decomposition spaces which is generated by a *single* prototype function in the same structured way as proposed in the present paper. Furthermore, Nielsen then uses a distortion argument to show that one can also obtain a structured Banach frame with a *single, compactly supported* generator. Hence, at a first glance, it might seem that all results of the present paper are already contained in [62]. This, however, is *not* true for the following reasons:

- As already observed above, the coverings considered in [62] and [63] are quite restricted. They have to be of the form $\mathcal{Q} = (Q_k)_{k \in \mathbb{Z}^d} = (\mathcal{B}_A(\xi_k, \varrho \cdot h(\xi_k)))_{k \in \mathbb{Z}^d}$, where the balls $\mathcal{B}_A(\xi, r) = \{\zeta \in \mathbb{R}^d \mid |\zeta - \xi|_A < r\}$ are defined using the quasi-metric $|\cdot|_A$, which is determined by a suitable matrix A .

In particular, the setting considered in [62] does *neither* include the case of homogeneous or inhomogeneous Besov spaces, nor the case of shearlet smoothness spaces. To see this, note that [62, Proposition 3.6] does not impose any vanishing moment conditions on the prototype γ (which is called g in the notation of [62]). In fact, it is even required that $\widehat{\gamma}(0) \neq 0$. But it is folklore that the generator of an (inhomogeneous or homogeneous) wavelet frame for $L^2(\mathbb{R})$ has to satisfy certain vanishing moment conditions; the proof for homogeneous wavelet frames is given in [19, Theorem 3.3.1]. A proof of the corresponding statement for discrete cone-adapted shearlet frames is given in Appendix C.

In stark contrast, the theory developed in the present paper *is* able to handle Besov spaces (cf. Section 8), as well as shearlet smoothness spaces (cf. the companion paper [66]).

- Since a distortion argument is used to obtain a compactly supported Banach frame from a bandlimited frame, the choice of the generator γ in [62] is quite restricted; γ has to be close enough to the generator of the bandlimited frame.

In contrast, the assumptions imposed on γ in the present paper are quite mild. In most concrete cases (in particular for α -modulation spaces, Besov spaces and shearlet smoothness spaces), the conditions reduce to suitable smoothness, decay and vanishing moment criteria, in conjunction with a certain nonvanishing condition for the Fourier transform $\widehat{\gamma}$.

- In the present paper, we also consider the decomposition spaces $\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$ where a *weighted* Lebesgue space $L_v^p(\mathbb{R}^d)$ is used. In contrast, [62] only considers the *unweighted* case.

We remark however that [62] jointly considers Triebel-Lizorkin type, as well as Besov type decomposition spaces. In contrast, at least in its present state, the approach developed in this paper only applies to the Besov type decomposition spaces.

Finally, we mention the recent paper [64] in which Ottosen and Nielsen take the “reverse” of the usual approach: Instead of starting with a given function space X and then constructing Banach frames or atomic decompositions for this space, the authors start with a given **painless nonstationary Gabor frame** $(h_{i,k})_{i,k \in \mathbb{Z}^d}$ satisfying

$$h_{i,k} = L_{a_i \cdot k} h_i \quad \text{and} \quad \text{supp } \widehat{h}_i \subset [0, a_i^{-1}]^d + b_i \quad \text{for certain} \quad a_i > 0 \text{ and } b_i \in \mathbb{R}^d.$$

Under suitable assumptions on the (slightly enlarged) covering

$$\mathcal{Q} = (Q_i)_{i \in \mathbb{Z}^d} \quad \text{with} \quad Q_i = a_i^{-1} \cdot (-\delta, 1 + \delta)^d + b_i,$$

Ottosen and Nielsen then show that the renormalized family $(h_{i,k}^{(p)})_{i,k \in \mathbb{Z}^d}$ defined by $h_{i,k}^{(p)} = a_i^{\frac{1}{p}-\frac{1}{2}} \cdot h_{i,k}$ forms a Banach frame for the decomposition space $\mathcal{D}(\mathcal{Q}, L^p, \ell_{\omega_s}^q)$, where $\omega_i = 1 + \|\xi_i\|^2$ for suitable $\xi_i \in Q_i$. In addition, it is shown in [64, Theorem 6.1] for $p, q \in (0, \infty)$ that every $f \in \mathcal{D}(\mathcal{Q}, L^p, \ell_{\omega_s}^q)$ admits an expansion of the form

$$f = \sum_{i,k \in \mathbb{Z}^d} \langle h, h_{i,k} \rangle \cdot \tilde{h}_{i,k}, \quad (1.8)$$

where $(\tilde{h}_{i,k})_{i,k \in \mathbb{Z}^d}$ is the canonical dual frame of the nonstationary Gabor frame $(h_{i,k})_{i,k \in \mathbb{Z}^d}$.

In summary, the paper [64] starts with a given painless nonstationary Gabor frame and then shows that the space of analysis-sparse signals w.r.t. the frame coincides with a suitably defined decomposition space. Note that the painless nonstationary Gabor frames are always bandlimited. Using the theory developed in this paper, it should be possible (perhaps with the cost of changing the sampling density in comparison to the original frame) to show similar results for nonstationary Gabor frames with *compactly supported* generators. Furthermore, while the results in [64] only show that each $f \in \mathcal{D}(\mathcal{Q}, L^p, \ell_w^q)$ admits a sparse expansion in terms of the *dual frame* $(\tilde{h}_{i,k})_{i,k \in \mathbb{Z}^d}$, our results would yield a sparse expansion in terms of the frame itself, so that analysis sparsity is equivalent to synthesis sparsity.

1.2. Notation and conventions. We write $\mathbb{N} = \mathbb{Z}_{\geq 1}$ for the set of **natural numbers** and $\mathbb{N}_0 = \mathbb{Z}_{\geq 0}$ for the set of natural numbers including 0. For a matrix $A \in \mathbb{C}^{d \times d}$, we denote by A^T the transpose of A . The norm $\|A\|$ of A is the usual **operator norm** of A , acting on \mathbb{R}^d equipped with the usual euclidean norm $|\cdot| = \|\cdot\|_2$. The **open euclidean ball** of radius $r > 0$ around $x \in \mathbb{R}^d$ is denoted by $B_r(x)$. For a linear (bounded) operator $T : X \rightarrow Y$ between (quasi)-normed spaces X, Y , we denote the **operator norm** of T by

$$\|T\| := \|T\|_{X \rightarrow Y} := \sup_{\|x\|_X \leq 1} \|Tx\|_Y.$$

For an arbitrary set M , we let $|M| \in \mathbb{N}_0 \cup \{\infty\}$ denote the number of elements of the set. For $n \in \mathbb{N}_0 = \mathbb{Z}_{\geq 0}$, we write $\underline{n} := \{1, \dots, n\}$; in particular, $\underline{0} = \emptyset$. For the **closure** of a subset M of some topological space, we write \overline{M} .

The d -dimensional **Lebesgue measure** of a (measurable) set $M \subset \mathbb{R}^d$ is denoted by $\lambda(M)$ or by $\lambda_d(M)$. Furthermore, for $M \subset \mathbb{R}^d$, we define the **indicator function** (or **characteristic function**) $\mathbb{1}_M$ of the set M by

$$\mathbb{1}_M : \mathbb{R}^d \rightarrow \{0, 1\}, x \mapsto \begin{cases} 1, & \text{if } x \in M, \\ 0, & \text{otherwise.} \end{cases}$$

For two subsets $A, B \subset \mathbb{R}^d$, we define the **Minkowski sum** and the **Minkowski difference** of A, B by

$$A + B := \{a + b \mid a \in A, b \in B\} \quad \text{and} \quad A - B := \{a - b \mid a \in A, b \in B\}.$$

The Minkowski difference $A - B$ should be distinguished from the **set-theoretic difference** $A \setminus B = \{a \in A \mid a \notin B\}$.

The **translation** and **modulation** of a function $f : \mathbb{R}^d \rightarrow \mathbb{C}^k$ by $x \in \mathbb{R}^d$ or $\xi \in \mathbb{R}^d$ are, respectively, denoted by

$$L_x f : \mathbb{R}^d \rightarrow \mathbb{C}^k, y \mapsto f(y - x), \quad \text{and} \quad M_\xi f : \mathbb{R}^d \rightarrow \mathbb{C}^k, y \mapsto e^{2\pi i \langle \xi, y \rangle} f(y).$$

For the **Fourier transform**, we use the convention $\widehat{f}(\xi) := (\mathcal{F}f)(\xi) := \int_{\mathbb{R}^d} f(x) \cdot e^{-2\pi i \langle x, \xi \rangle} dx$ for $f \in L^1(\mathbb{R}^d)$. It is well-known that the Fourier transform extends to a unitary automorphism $\mathcal{F} : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$. The inverse of this map is the continuous extension of the inverse Fourier transform, given by $(\mathcal{F}^{-1}f)(x) = \int_{\mathbb{R}^d} f(\xi) e^{2\pi i \langle x, \xi \rangle} d\xi$ for $f \in L^1(\mathbb{R}^d)$. We will make frequent use of the space $\mathcal{S}(\mathbb{R}^d)$ of **Schwartz functions** and its dual space $\mathcal{S}'(\mathbb{R}^d)$, the space of **tempered distributions**. For more details on these spaces, we refer to [29, Section 9]; in particular, we note that the Fourier transform restricts to a linear homeomorphism $\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$; by duality, we can thus define $\mathcal{F} : \mathcal{S}'(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)$ by $\mathcal{F}\varphi = \varphi \circ \mathcal{F}$ for $\varphi \in \mathcal{S}'(\mathbb{R}^d)$.

Given an open subset $U \subset \mathbb{R}^d$, we let $\mathcal{D}'(U)$ denote the space of **distributions** on U , i.e., the topological dual space of $C_c^\infty(U)$. For the precise definition of the topology on $C_c^\infty(U)$, we refer to [70, Chapter 6]. We remark that the dual pairings $\langle \cdot, \cdot \rangle_{\mathcal{D}', \mathcal{D}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{S}', \mathcal{S}}$ are always taken to be *bilinear* instead of sesquilinear.

We write $v_d := \lambda_d(B_1(0))$ for the d -dimensional Lebesgue measure of the euclidean unit ball. An easy, but sometimes useful estimate is that $v_d \leq 2^d$, since $B_1(0) \subset [-1, 1]^d$. Furthermore, we let $s_d := \mathcal{H}_{d-1}(S^{d-1})$ denote

the surface measure of the unit sphere. It is well-known that $s_d = d \cdot v_d \leq d \cdot 2^d \leq 2^{2d}$, since $d \leq 2^d$. Finally, we have $B_1^{\|\cdot\|_\infty}(0) \subset B_{\sqrt{d}}(0)$ and thus $2^d = \lambda(B_1^{\|\cdot\|_\infty}(0)) \leq \lambda(B_{\sqrt{d}}(0)) = v_d \cdot d^{d/2}$, which implies $v_d \geq (2/\sqrt{d})^d$.

The constant s_d be important for us due to the following: For $p \in (0, \infty)$ and $N > d/p$, we get using polar coordinates that

$$\begin{aligned} \|(1 + |\bullet|)^{-N}\|_{L^p}^p &= \int_{\mathbb{R}^d} (1 + |x|)^{-Np} dx = \int_0^\infty r^{d-1} \int_{S^{d-1}} (1 + |r\xi|)^{-Np} d\mathcal{H}_{d-1}(\xi) dr \\ &= \mathcal{H}_{d-1}(S^{d-1}) \cdot \int_0^\infty r^{d-1} \cdot (1 + r)^{-Np} dr \\ &\leq s_d \cdot \int_0^\infty (1 + r)^{d-Np-1} dr \\ &\quad (\text{since } d-Np < 0) = s_d \cdot \frac{(1 + r)^{d-Np}}{d - Np} \Big|_0^\infty = \frac{s_d}{Np - d}, \end{aligned}$$

and hence

$$\|(1 + |\bullet|)^{-N}\|_{L^p} \leq \left(\frac{1}{p} \cdot \frac{s_d}{N - \frac{d}{p}} \right)^{1/p} \quad \forall N > d/p, \quad (1.9)$$

which also remains valid (with the interpretation $x^0 = 1$ for arbitrary $x \geq 0$) for $p = \infty$.

1.3. Definition of decomposition spaces and standing assumptions. For the whole paper, we fix a **semi-structured admissible covering** $\mathcal{Q} = (Q_i)_{i \in I}$ of an open subset $\mathcal{O} \subset \mathbb{R}^d$. Precisely this means that for each $i \in I$ there is a measurable subset $Q'_i \subset \mathbb{R}^d$, an invertible linear map $T_i \in \text{GL}(\mathbb{R}^d)$ and a translation $b_i \in \mathbb{R}^d$ such that $Q_i = S_i Q'_i = T_i Q'_i + b_i$ for the affine transformation $S_i : \mathbb{R}^d \rightarrow \mathbb{R}^d, \xi \mapsto T_i \xi + b_i$ and such that the following properties are fulfilled:

- (1) \mathcal{Q} covers \mathcal{O} , i.e., $\mathcal{O} = \bigcup_{i \in I} Q_i$.
- (2) \mathcal{Q} is **admissible**, i.e., we have $|i^*| \leq N_{\mathcal{Q}} < \infty$ for all $i \in I$, where
$$i^* := \{\ell \in I \mid Q_\ell \cap Q_i \neq \emptyset\}. \quad (1.10)$$
- (3) There is some $R_{\mathcal{Q}} > 0$ satisfying $Q'_i \subset \overline{B_{R_{\mathcal{Q}}}}(0)$ for all $i \in I$.
- (4) There is some $C_{\mathcal{Q}} > 0$ satisfying $\|T_i^{-1} T_\ell\| \leq C_{\mathcal{Q}}$ for all $i \in I$ and all $\ell \in i^*$.

The most common form of decomposition spaces uses a (quasi)-norm of the form $\left\| (\|\mathcal{F}^{-1}(\varphi_i \cdot \widehat{g})\|_{L^p})_{i \in I} \right\|_{\ell_w^q}$, i.e., the frequency-localized pieces $g_i = \mathcal{F}^{-1}(\varphi_i \cdot \widehat{g})$ of g are measured in $L^p(\mathbb{R}^d)$. To achieve even greater flexibility, we will allow weighted Lebesgue spaces of the form $L_v^p(\mathbb{R}^d)$ instead of $L^p(\mathbb{R}^d)$. Here, we write

$$L_v^p(\mathbb{R}^d) := \{f : \mathbb{R}^d \rightarrow \mathbb{C} \mid f \text{ measurable and } v \cdot f \in L^p(\mathbb{R}^d)\},$$

equipped with the natural (quasi)-norm $\|f\|_{L_v^p} := \|v \cdot f\|_{L^p}$. In order to still obtain reasonable spaces and results, we assume the following:

- (5) The weights $v, v_0 : \mathbb{R}^d \rightarrow (0, \infty)$ are measurable and satisfy the following:
 - (a) $v_0 \geq 1$ and v_0 is symmetric, i.e., $v_0(-x) = v_0(x)$ for all $x \in \mathbb{R}^d$.
 - (b) v_0 is submultiplicative, i.e., $v_0(x + y) \leq v_0(x) \cdot v_0(y)$ for all $x, y \in \mathbb{R}^d$.
 - (c) v is v_0 -moderate, i.e., $v(x + y) \leq v(x) \cdot v_0(y)$ for all $x, y \in \mathbb{R}^d$.
 - (d) There is some $K \geq 0$ and some $\Omega_1 \geq 1$ satisfying $v_0(x) \leq \Omega_1 \cdot (1 + |x|)^K$ for all $x \in \mathbb{R}^d$.
 - (e) The constant K from the previous step satisfies $K = 0$ or there is a constant $\Omega_0 \geq 1$ satisfying $\|T_i^{-1}\| \leq \Omega_0$ for all $i \in I$.
- (6) There is a \mathcal{Q} - v_0 -**BAPU** (bounded admissible partition of unity) $\Phi = (\varphi_i)_{i \in I}$ for \mathcal{Q} , which means that:
 - (a) $\varphi_i \in C_c^\infty(\mathcal{O})$ for all $i \in I$ and furthermore $\varphi_i \equiv 0$ on $\mathcal{O} \setminus Q_i$.
 - (b) $\sum_{i \in I} \varphi_i \equiv 1$ on \mathcal{O} .
 - (c) For each $p \in (0, \infty]$, the following expression (then a constant) is finite:
$$C_{\mathcal{Q}, \Phi, v_0, p} := \sup_{i \in I} \left[|\det T_i|^{\max\{\frac{1}{p}, 1\}-1} \cdot \|\mathcal{F}^{-1} \varphi_i\|_{L_{v_0}^{\min\{1, p\}}} \right].$$

²One can always assume $v_0 \geq 1$ without loss of generality, since all properties of v_0 (including submultiplicativity) are also fulfilled for $\tilde{v}_0 := 1 + v_0 \geq 1$, where possibly Ω_1 has to be enlarged, since $\tilde{v}_0(x) \leq 1 + \Omega_1 \cdot (1 + |x|)^K \leq (1 + \Omega_1) \cdot (1 + |x|)^K$. Likewise, by switching to $\tilde{v}_0(x) := v_0(x) + v_0(-x)$, one can always assume v_0 to be symmetric.

Clearly, if one chooses $K = 0$ and $v = v_0 \equiv 1$, then one obtains the usual decomposition spaces, as considered e.g. in [9, 76, 77, 78]. This will be the most common case. Note that in this case, we do *not* need to assume $\|T_i^{-1}\| \leq \Omega_0$ for all $i \in I$, i.e., the covering \mathcal{Q} can be very general.

We observe for later use that the preceding assumptions imply

$$(1 + |x|)^K \leq \Omega_0^K \cdot (1 + |T_i^T x|)^K \quad \forall x \in \mathbb{R}^d. \quad (1.11)$$

Indeed, in case of $K = 0$, this is trivial. In case of $K > 0$, our assumptions imply

$$|x| = |T_i^{-T} T_i^T x| \leq \|T_i^{-T}\| \cdot |T_i^T x| = \|T_i^{-1}\| \cdot |T_i^T x| \leq \Omega_0 \cdot |T_i^T x|$$

and hence $1 + |x| \leq 1 + \Omega_0 \cdot |T_i^T x| \leq \Omega_0 \cdot (1 + |T_i^T x|)$, where the last step used that $\Omega_0 \geq 1$. This easily shows that equation (1.11) remains valid also for $K > 0$.

Finally, we observe for later use the convolution relation $L_{v_0}^1(\mathbb{R}^d) * L_v^p(\mathbb{R}^d) \hookrightarrow L_v^p(\mathbb{R}^d)$ for $p \in [1, \infty]$. Indeed, we have

$$\begin{aligned} v(x) \cdot |(f * g)(x)| &\leq v(x) \cdot \int_{\mathbb{R}^d} |f(y)| \cdot |g(x-y)| \, dy \\ &\quad (\text{since } v(x) = v(x-y+y) \leq v(x-y) \cdot v_0(y)) \leq \int_{\mathbb{R}^d} |(v_0 \cdot f)(y)| \cdot |(v \cdot g)(x-y)| \, dy, \end{aligned}$$

so that Minkowski's inequality for integrals (cf. [29, Theorem (6.19)]), together with the isometric translation invariance of $L^p(\mathbb{R}^d)$, yields

$$\begin{aligned} \|f * g\|_{L_v^p} &= \|v \cdot (f * g)\|_{L^p} \leq \left\| x \mapsto \int_{\mathbb{R}^d} |(v_0 \cdot f)(y)| \cdot |(v \cdot g)(x-y)| \, dy \right\|_{L^p} \\ &\leq \int_{\mathbb{R}^d} \|x \mapsto |(v_0 \cdot f)(y)| \cdot |(v \cdot g)(x-y)|\|_{L^p} \, dy \\ &= \int_{\mathbb{R}^d} |(v_0 \cdot f)(y)| \, dy \cdot \|v \cdot g\|_{L^p} = \|f\|_{L_{v_0}^1} \cdot \|g\|_{L_v^p} < \infty. \end{aligned} \quad (1.12)$$

We will call this estimate the **weighted Young inequality**. In particular, it shows that $(|f| * |g|)(x) < \infty$ for almost all $x \in \mathbb{R}^d$.

Given a \mathcal{Q} - v_0 -BAPU $\Phi = (\varphi_i)_{i \in I}$, we define the **clustered version** of Φ as $\Phi^* = (\varphi_i^*)_{i \in I}$, where $\varphi_i^* := \sum_{\ell \in i^*} \varphi_\ell$. Because of $\sum_{i \in I} \varphi_i \equiv 1$ on $\mathcal{O} \supset Q_i$ and since $\varphi_\ell \equiv 0$ on Q_i for all $\ell \in I \setminus i^*$, it is not hard to see $\varphi_i^* \equiv 1$ on Q_i , a property which we will use frequently. In particular, since $\varphi_i^* \in C_c^\infty(\mathcal{O})$ as a finite sum of elements of $C_c^\infty(\mathcal{O})$, we see that $\overline{Q_i} \subset \mathcal{O}$ is compact.

Next, we fix a **\mathcal{Q} -moderate weight** $w = (w_i)_{i \in I}$, which means that $w_i \in (0, \infty)$ for each $i \in I$ and that there is a constant $C_{\mathcal{Q}, w} > 0$ such that

$$w_i \leq C_{\mathcal{Q}, w} \cdot w_\ell \quad \forall i \in I \text{ and } \ell \in i^*. \quad (1.13)$$

Under these assumptions, it follows from [77, Lemma 4.13] that the **\mathcal{Q} -clustering map**

$$\Gamma_{\mathcal{Q}} : \ell_w^q(I) \rightarrow \ell_w^q(I), (c_i)_{i \in I} \mapsto (c_i^*)_{i \in I} \quad \text{with} \quad c_i^* := \sum_{\ell \in i^*} c_\ell \quad (1.14)$$

is well-defined and bounded with

$$\|\Gamma_{\mathcal{Q}}\| \leq C_{\mathcal{Q}, w} \cdot N_{\mathcal{Q}}^{1+\frac{1}{q}}. \quad (1.15)$$

Here, the **weighted sequence space** $\ell_w^q(I)$ is given by

$$\ell_w^q(I) := \{c = (c_i)_{i \in I} \in \mathbb{C}^I \mid \|c\|_{\ell_w^q} := \|(w_i \cdot c_i)_{i \in I}\|_{\ell^q} < \infty\},$$

for arbitrary $q \in (0, \infty]$.

Given all of these assumptions, we define for $p, q \in (0, \infty]$ the **Fourier-side decomposition space** associated to \mathcal{Q} and the parameters p, q, v, w as

$$\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L_v^p, \ell_w^q) := \left\{ f \in \mathcal{D}'(\mathcal{O}) \mid \|f\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L_v^p, \ell_w^q)} := \left\| \left(\|\mathcal{F}^{-1}(\varphi_i f)\|_{L_v^p} \right)_{i \in I} \right\|_{\ell_w^q} < \infty \right\}.$$

Finally, we set $Z(\mathcal{O}) := \mathcal{F}(C_c^\infty(\mathcal{O}))$, equipped with the unique topology which makes the Fourier transform $\mathcal{F} : C_c^\infty(\mathcal{O}) \rightarrow Z(\mathcal{O})$ a topological isomorphism. Then, with $Z'(\mathcal{O})$ denoting the topological dual space of $Z(\mathcal{O})$, we define the (space-side) **decomposition space** associated to \mathcal{Q} and the parameters p, q, v, w as

$$\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q) := \left\{ g \in Z'(\mathcal{O}) \mid \|g\|_{\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)} := \|\widehat{g}\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L_v^p, \ell_w^q)} < \infty \right\},$$

where $\mathcal{F}g := \widehat{g} := g \circ \mathcal{F} \in \mathcal{D}'(\mathcal{O})$ for $g \in Z'(\mathcal{O})$. It is not hard to see that the Fourier transform $\mathcal{F} : Z'(\mathcal{O}) \rightarrow \mathcal{D}'(\mathcal{O})$ restricts to an isometric isomorphism $\mathcal{F} : \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q) \rightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L_v^p, \ell_w^q)$ and that we have $Z'(\mathcal{O}) = \mathcal{F}^{-1}(\mathcal{D}'(\mathcal{O}))$.

For an explanation for the choice of the reservoirs $\mathcal{D}'(\mathcal{O})$ and $Z'(\mathcal{O})$, we refer to [77, Remark 3.13]. Finally, we mention that [77, Section 8] provides a convenient criterion which ensures that each $f \in \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$ extends to a tempered distribution. In particular, if $v \gtrsim 1$, then clearly $\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q) \hookrightarrow \mathcal{D}(\mathcal{Q}, L^p, \ell_w^q)$. Hence, if the previously mentioned criterion is fulfilled and if $\mathcal{O} = \mathbb{R}^d$, we have (up to trivial identifications) that

$$\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q) = \left\{ g \in \mathcal{S}'(\mathbb{R}^d) \mid \|g\|_{\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)} = \left\| \left(\|\mathcal{F}^{-1}(\varphi_i \widehat{g})\|_{L_v^p} \right)_{i \in I} \right\|_{\ell_w^q} < \infty \right\}.$$

We remark that the usual papers treating general decomposition spaces (for general $p, q \in (0, \infty]$) do usually only consider the case $v \equiv 1$. Hence, it is not entirely clear that the spaces defined here are indeed well-defined (Quasi)-Banach spaces for $v \neq 1$. We will see below (cf. Proposition 2.24 and Lemma 5.5) that this is indeed the case.

1.4. Structure of the paper. The theory of decomposition spaces is highly dependent on convolutions, since the very definition of the norm involves quantities of the form

$$\|\mathcal{F}^{-1}(\varphi_i \cdot \widehat{g})\|_{L_v^p} = \|(\mathcal{F}^{-1}\varphi_i) * g\|_{L_v^p}.$$

For the range $p \in [1, \infty]$, Young's inequality $L^1 * L^p \hookrightarrow L^p$ is usually sufficient to handle such convolutions. But in the range $p \in (0, 1)$, Young's inequality breaks down completely. For the usual theory of decomposition spaces, one instead invokes the convolution relation

$$\|f * g\|_{L^p} \leq C_{p,d} \cdot R^{d(\frac{1}{p}-1)} \cdot \|f\|_{L^p} \cdot \|g\|_{L^p} \quad \text{assuming} \quad \text{supp } \widehat{f} \subset B_R(\xi_1) \text{ and } \text{supp } \widehat{g} \subset B_R(\xi_2)$$

for certain $\xi_1, \xi_2 \in \mathbb{R}^d$. Note though that this convolution relation only applies to *band-limited* functions. But since we are interested in characterizations of decomposition spaces using (possibly) *compactly supported* functions, this is not of much use to us.

To overcome this problem, we will invoke the theory of the **Wiener amalgam spaces** $W_Q(L^\infty, L_v^p)$ which were originally introduced by Feichtinger[22]. The main idea is to associate to a (measurable) function $f : \mathbb{R}^d \rightarrow \mathbb{C}$ the local maximal function

$$M_Q f : \mathbb{R}^d \rightarrow [0, \infty], x \mapsto \|\mathbf{1}_{x+Q} \cdot f\|_{L^\infty}$$

and to define the Wiener amalgam (quasi)-norm of f as $\|f\|_{W_Q(L^\infty, L_v^p)} = \|M_Q f\|_{L_v^p}$. Broadly speaking, functions in $W_Q(L^\infty, L_v^p)$ are locally in L^∞ and globally in L_v^p . For brevity, we will simply write $W_Q(L_v^p) := W_Q(L^\infty, L_v^p)$. For these spaces, convolution relations are known, cf. [68] and [76, Section 2.3]. For our purposes, however, these results are not sufficient: They establish estimates of the form

$$\|f * g\|_{W_Q(L_v^p)} \leq C_{p,Q,v} \cdot \|f\|_{W_Q(L_{v_0}^p)} \cdot \|g\|_{W_Q(L_v^p)},$$

where the constant $C_{p,Q,v}$ depends heavily—and *in an unspecified way*—on Q . But for our purposes, we will consider the spaces $W_{T_i^{-T}[-1,1]^d}(L^\infty, L_v^p)$ where $i \in I$ varies; see for example equation (1.4). Then, we will need estimates of the form

$$\|f * g\|_{W_{T_i^{-T}[-1,1]^d}(L_v^p)} \leq C_{i,j,\ell,p,v} \cdot \|f\|_{W_{T_j^{-T}[-1,1]^d}(L_{v_0}^p)} \cdot \|g\|_{W_{T_\ell^{-T}[-1,1]^d}(L_v^p)},$$

with precise control on the constant $C_{i,j,\ell,p,v}$. Hence, in Section 2, we redevelop parts of the theory of Wiener amalgam spaces, paying close attention to the dependence of certain constants on the base-set Q .

Next, in Section 3, we derive assumptions on the prototype function γ which ensure that the norm equivalences given in equations (1.3) and (1.4) are true. More precisely, we will show that the map

$$\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q) \rightarrow \ell_w^q([V_i]_{i \in I}), g \mapsto \left(\gamma^{(i)} * g \right)_{i \in I}$$

forms a Banach frame, where $V_i := L_v^p(\mathbb{R}^d)$ in case of $p \in [1, \infty]$ and $V_i := W_{T_i^{-T}[-1,1]^d}(L_v^p)$ in case of $p \in (0, 1)$ and where finally

$$\ell_w^q([V_i]_{i \in I}) = \left\{ (g_i)_{i \in I} \mid (\forall i \in I : g_i \in V_i) \text{ and } (\|g_i\|_{V_i})_{i \in I} \in \ell_w^q(I) \right\}.$$

Part of the problem is to explain how the convolution $\gamma^{(i)} * g$ can be interpreted, especially in case of $\mathcal{O} \subsetneq \mathbb{R}^d$, since then each element g of the decomposition space $\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$ is the inverse Fourier transform of the distribution $\widehat{g} \in \mathcal{D}'(\mathcal{O})$, so that it is not obvious how $\gamma^{(i)} * g$ can be understood.

In Section 4, we further discretize the Banach frame $(\gamma^{(i)})_{i \in I}$ from above: Under slightly more strict assumptions on γ than before, we will be able to show that the family $\Psi_\delta = \left(L_{\delta \cdot T_i^{-T} k} \widetilde{\gamma^{[i]}} \right)_{i \in I, k \in \mathbb{Z}^d}$ forms a **Banach frame** for $\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$, once $\delta > 0$ is chosen small enough. Our proof technique is similar to that of coorbit theory: We use the partition of unity $(\varphi_i)_{i \in I}$ associated to the covering \mathcal{Q} to obtain a kind of reproduction formula, which we then discretize. The details, however, are quite technical.

Next, in Section 5 we establish the dual statement that the family $\left(L_{\delta \cdot T_i^{-T} k} \gamma^{[i]} \right)_{i \in I, k \in \mathbb{Z}^d}$ forms an **atomic decomposition** for $\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$. As above, this is based on a suitable discretization of a certain reproduction formula.

Finally, since the varying assumptions placed on the prototype γ are quite technical and hard to verify, Section 6 is devoted to a considerable simplification of these conditions. While not exactly straightforward to verify, these conditions can be verified in practice, where the degree of difficulty mainly depends on the given covering $\mathcal{Q} = (T_i Q'_i + b_i)_{i \in I}$.

As a litmus test of our theory, we show in Section 7 that it can be used to obtain compactly supported structured Banach frames and atomic decompositions for the α -modulation spaces $M_{\alpha, s}^{p, q}(\mathbb{R}^d)$, even for $p, q < 1$, thereby extending the state of the art. Furthermore, in Section 8, we show that our theory can be used to establish that certain compactly supported wavelet systems generate Banach frames and atomic decompositions for inhomogeneous Besov spaces.

We emphasize that we consider these two specific examples since they can be handled with reasonably low effort, but still indicate that—and how—the general theory can be filled with life for concrete special cases. The theory presented here certainly has more interesting and more novel applications, in particular to the theory of shearlets. But in order to keep the size of this paper somewhat manageable, we postpone these applications to the companion paper [66].

Credit where credit is due.

“[...] virtually all of our techniques already exist in some antecedent form. Nevertheless their particular combination here leads to new conclusions and to sharpened versions of known results. Moreover, our presentation reveals a[...] structure underlying a diverse range of topics in harmonic analysis.”

M. Frazier and B. Jawerth, [33, Page 36]

The results and proof techniques employed in this paper were heavily inspired by a number of earlier results:

The first impulse for writing this paper was caused by my reading of the paper [52]. In that paper, the author characterizes the existence of embeddings between α -modulation spaces and Sobolev spaces. As an intermediate result, he also proves

$$\|g\|_{M_{\alpha, s}^{p, q}} \asymp \|(\|\gamma^{(i)} * g\|_{L^p})_{i \in \mathbb{Z}^d}\|_{\ell_{(\langle k \rangle)^s}^q} \quad (1.16)$$

for arbitrary $p, q \in (0, \infty]$, $\alpha \in [0, 1)$ and $s \in \mathbb{R}$, as well as $g \in M_{\alpha, s}^{p, q}(\mathbb{R}^d)$, if the prototype function $\gamma \in \mathcal{S}(\mathbb{R}^d)$ is chosen suitably. Here, the functions $\gamma^{(i)}$ for $i \in \mathbb{Z}^d$ are formed from γ as described before equation (1.2), where $\mathcal{Q} = \mathcal{Q}^{(\alpha)} = \left(\langle k \rangle^{\frac{\alpha}{1-\alpha}} \cdot B_R(0) + \langle k \rangle^{\frac{\alpha}{1-\alpha}} k \right)_{k \in \mathbb{Z}^d}$ is the usual covering used to define α -modulation spaces; see also Section 7. Note that—at least for $p \in [1, \infty]$ —this result is a special case of the results about semi-discrete Banach frames from Section 3. Specifically, the paper [52] caused me to investigate whether a norm characterization as in equation (1.16) was also possible in the more general setting of (essentially) arbitrary decomposition spaces and not only for α -modulation spaces. In particular, it caused me to consider the structured families of the form $(\gamma^{(i)})_{i \in I}$ with $\gamma^{(i)} = |\det T_i| \cdot M_{b_i}[\gamma \circ T_i^T]$, where $\mathcal{Q} = (T_i Q'_i + b_i)_{i \in I}$.

Furthermore, an investigation of the proofs in [52] lead me to consider assumptions similar to those stated in Assumption 3.1 below. Specifically, it caused me to impose boundedness of the operator associated to the infinite matrix $\left(\left\|\mathcal{F}^{-1}\left(\varphi_i \cdot \widehat{\gamma^{(j)}}\right)\right\|_{L^1}\right)_{j,i \in I}$. In summary, at least for the case $p \in [1, \infty]$, the results about semi-discrete Banach frames for decomposition spaces in this paper (cf. Section 3) can be seen as a slight generalization of the results in [52].

For the case $p \in (0, 1)$, however, I was not able to adapt the techniques used in [52] to the general setting of decomposition spaces. In fact, for $p \in (0, 1)$, the results derived in [52] differ from those in Section 3: While the characterization from [52] (cf. equation (1.16)) considers the usual L^p norm of the convolutions $\gamma^{(i)} * f$, in Section 3 we show for $p \in (0, 1)$ that

$$\|g\|_{\mathcal{D}(\mathcal{Q}, L^p, \ell_w^q)} \asymp \left\| \left(\|\gamma^{(i)} * g\|_{W_{T_i - T_{[-1,1]^d}}(L^p)} \right)_{i \in I} \right\|_{\ell_w^q},$$

i.e., we use **Wiener amalgam spaces** instead of the spaces L^p themselves.

Here, again, I was inspired heavily by earlier results: The main limitation of the spaces $L^p(\mathbb{R}^d)$ for $p \in (0, 1)$ in the present setting is that there are no meaningful convolution relations for them, partly since we do not even have $L^p(\mathbb{R}^d) \hookrightarrow L_{\text{loc}}^1(\mathbb{R}^d)$. Luckily, Holger Rauhut[67] had already observed—while generalizing coorbit theory [25, 26, 27, 44] to the setting of Quasi-Banach spaces—that these limitations can be avoided by considering the Wiener amalgam spaces $W_Q(L^p)$ instead of L^p itself. Rauhut had also already developed associated convolution relations[68] for these spaces. Of course, all of this was based on the original invention of Wiener Amalgam spaces which is due to Hans Feichtinger[21, 22].

All in all, given these earlier papers, it was natural to consider Wiener amalgam spaces. The (as far as I know) novel idea was to consider these Wiener amalgam spaces $W_Q(L^p)$ with a definite choice of the base set Q , which was allowed to heavily vary with $i \in I$. Further, it seems to be a new (or at least not completely well-known) fact that suitably bandlimited L^p functions automatically belong to $W_Q(L^p)$, where this statement comes with a precise estimate for the Wiener amalgam norm in terms of Q and the Fourier support of the function.

At this point, I had managed to generalize the results about semi-discrete Banach frames developed in [52] to the setting of general decomposition spaces. One of my main goals, however, was a better understanding of the approximation theoretic properties of discrete, cone-adapted shearlet systems. To achieve this, a further discretization of these *semi*-discrete Banach frames was necessary. The inspiration for treating this additional discretization step came from the theory of coorbit spaces as developed by Feichtinger and Gröchenig[25, 26, 27, 44] and also (in more generalized form) by Rauhut, Fornasier and Ullrich[67, 31, 69]. The underlying important idea of coorbit theory is to transfer the study of certain function spaces via a suitable transform to the study of certain Banach spaces which have a **reproducing property**. Formally, one employs the so-called voice transform V to establish an isomorphism between the coorbit space $\text{Co}(Y)$ and its image $Z := V[\text{Co}(Y)]$ under the voice transform. The crucial property of the space Z is that we have the **reproducing formula**

$$F = F * G \quad \forall F \in Z$$

for a suitable kernel G . In fact, in the setting of generalized coorbit theory, the convolution with G needs to be replaced by a more general integral operator.

If the kernel G is regular enough, the reproducing formula allows to show that a sufficiently dense sampling of $F \in Z$ suffices to reconstruct F uniquely. Proving this is based on a (suitable) notion of the **oscillation** of a function. This sampling result can then be transferred to the coorbit space $\text{Co}(Y)$ to obtain Banach frames and atomic decompositions. Similar techniques are also used in [2].

The new contribution was thus to derive a suitable reproducing formula in the general setting of decomposition spaces, cf. Lemma 4.6. Once this was established, existing ideas and techniques could be used to obtain the desired discrete Banach frames and atomic decompositions. We remark, however, that the established reproducing formula for decomposition spaces is highly nontrivial.

In total, the present paper would not have been possible without inspiration from existing results, concepts and techniques (Wiener amalgam spaces and their convolution relations, oscillation of a function, semi-discrete Banach frames for α -modulation spaces, etc.). The contribution of the paper is that these results and techniques are combined and refined to achieve novel and nontrivial results which—due to their generality—apply in a wide variety of settings.

A comment on constants. Instead of using only implied constants of the form $C = C(d, p, \mathcal{Q}, \dots)$, in this paper we try to provide explicit constants whenever possible. In principle, this allows one e.g. to determine an *explicit* $\delta_0 > 0$ such that the family Ψ_δ defined in equation (1.5) yields a Banach frame for the decomposition space under consideration for $0 < \delta \leq \delta_0$. We make no effort, however, to produce the optimal (or even good) constants. Occasionally, we even enlarge appearing constants just to make the expressions for the constants in question more optically pleasing (i.e., shorter). Due to these reasons, the resulting sampling density δ_0 will probably be of size $\delta_0 \approx 2^{-1000}$ or even smaller.

Thus, our leading philosophy is that *an arbitrarily bad explicit constant is still (much) better than an implicit constant which one does not know at all.*

2. CONVOLUTION IN L^p , $p \in (0, 1)$ AND WIENER AMALGAM SPACES

The well-known Young inequality $\|f * g\|_{L^p} \leq \|f\|_{L^1} \cdot \|g\|_{L^p}$ fails for $p \in (0, 1)$, cf. [77, Example 3.1]. One can solve this in two ways: The first way is given in [73, Proposition 1.5.1], where it is shown that

$$\|f * g\|_{L^p} \lesssim \|f\|_{L^p} \cdot \|g\|_{L^p}$$

if f and g are both bandlimited. This theorem, however, has two disadvantages:

- The restriction to bandlimited f, g is rather severe; in particular in our present setting, since we are interested in compactly supported functions, which can never be bandlimited.
- The implicit constant in the estimate above depends in a nontrivial way on the frequency supports of f, g .

To overcome these limitations, we will develop an improved theory of convolution for L^p , $p \in (0, 1)$ using the theory of **Wiener amalgam spaces**. As a special case, we will recover the estimate from above.

Before developing the theory, we remark that essentially everything mentioned in this section is already known in one form or another. In particular, Wiener amalgam spaces were originally invented by Feichtinger[21, 22] and later generalized to Quasi-Banach spaces by Rauhut[67]. The use of these spaces—and of the oscillation of a function—for obtaining Banach frames and atomic decompositions for certain spaces goes back to the theory of coorbit spaces[25, 26, 27, 44, 31, 69] and was also exploited in [2]. Therefore, no originality is claimed.

The usual treatments, however, mostly ignore or suppress the dependence of the Wiener amalgam spaces on the chosen unit neighborhood (see below for details), whereas this dependence is crucial for us. Hence, we provide full proofs.

2.1. Definition of Wiener amalgam spaces. All of the theory of Wiener amalgam spaces is centered around the notion of a certain maximal function:

Definition 2.1. (cf. [22], [51, Definition 2.2.2], [68] and [76, Definition 2.3.1]) Let $Q \subset \mathbb{R}^d$ be a Borel measurable unit neighborhood and let $f : \mathbb{R}^d \rightarrow \mathbb{C}^k$ be Borel measurable. We then define the **Q -maximal function** of f as

$$M_Q f : \mathbb{R}^d \rightarrow [0, \infty], x \mapsto \operatorname{ess\,sup}_{y \in x+Q} |f(y)| = \operatorname{ess\,sup}_{a \in Q} |f(x+a)| = \|L_{-x} f\|_{L^\infty(Q)}.$$

For a given $p \in (0, \infty]$ and a (measurable) weight $u : \mathbb{R}^d \rightarrow (0, \infty)$, we define the **Wiener amalgam space** with window Q , local component L^∞ and global component L_u^p as

$$W_Q^k(L_u^p) := W_Q^k(L^\infty, L_u^p) := \{f : \mathbb{R}^d \rightarrow \mathbb{C}^k \mid f \text{ measurable and } M_Q f \in L_u^p(\mathbb{R}^d)\},$$

with the natural (quasi)-norm $\|f\|_{W_Q^k(L_u^p)} := \|M_Q f\|_{L_u^p}$. In the most common case $k = 1$, we omit the exponent and write $W_Q(L_u^p)$ instead of $W_Q^1(L_u^p)$. ◀

Remark. • One can show for suitable weights u (and we will do so in Lemma 2.7) that the space $W_Q(L_u^p)$ is independent of the choice of the *bounded* measurable unit neighborhood $Q \subset \mathbb{R}^d$, with equivalent quasi-norms for different choices. Hence, Q is often suppressed in the literature dealing with Wiener amalgam spaces. For us, however, the precise choice of Q will be crucial, since we will choose $Q_i = T_i^{-T}[-1, 1]^d$, so that the sets Q_i vary wildly with $i \in I$. Since the constants appearing in the norm equivalences for different choices of Q depend heavily on the actual choices of Q , we will almost never use the equivalence for different choices of Q , or only in very carefully chosen ways.

- Note that $M_Q f$ is always a Borel measurable function. Indeed, since $L^1(\mathbb{R}^d)$ is separable, there is a countable dense family $(g_n)_{n \in \mathbb{N}}$ in $\Gamma := \{g \in L^1(\mathbb{R}^d) \mid g \geq 0 \text{ and } \|g\|_{L^1} \leq 1\}$. Then, we have for an arbitrary measurable function f that

$$\|f\|_{L^\infty} = \sup_{n \in \mathbb{N}} \int_{\mathbb{R}^d} g_n(x) \cdot |f(x)| \, dx. \quad (2.1)$$

For $f \in L^\infty(\mathbb{R}^d)$, this follows from the usual characterization of the L^∞ -norm by duality (cf. [29, Theorem 6.14]). In case of $\|f\|_{L^\infty} = \infty$, the same theorem shows that for each $M > 0$, there is some $g \in \Gamma$ satisfying $\int_{\mathbb{R}^d} g(x) \cdot |f(x)| dx \geq M$. But by density of the family $(g_n)_{n \in \mathbb{N}}$, there is then a sequence $(n_k)_{k \in \mathbb{N}}$ such that $g_{n_k} \rightarrow g$ in $L^1(\mathbb{R}^d)$. By switching to a subsequence, we can also assume $g_{n_k} \rightarrow g$ almost everywhere. Now, Fatou's Lemma yields

$$M \leq \int_{\mathbb{R}^d} g(x) \cdot |f(x)| dx = \int_{\mathbb{R}^d} \liminf_{k \rightarrow \infty} g_{n_k}(x) \cdot |f(x)| dx \leq \liminf_{k \rightarrow \infty} \int_{\mathbb{R}^d} g_{n_k}(x) \cdot |f(x)| dx.$$

Since $M > 0$ was arbitrary, this easily yields $\sup_{n \in \mathbb{N}} \int_{\mathbb{R}^d} g_n(x) \cdot |f(x)| dx = \infty = \|f\|_{L^\infty}$.

Now, as a consequence of equation (2.1), we get

$$\begin{aligned} (M_Q f)(x) &= \|\mathbb{1}_Q \cdot L_{-x} f\|_{L^\infty} \\ &= \sup_{n \in \mathbb{N}} \int_{\mathbb{R}^d} g_n(y) \cdot \mathbb{1}_Q(y) \cdot |(L_{-x} f)(y)| dy \\ &= \sup_{n \in \mathbb{N}} \int_{\mathbb{R}^d} g_n(y) \cdot \mathbb{1}_Q(y) \cdot |f(x+y)| dy. \end{aligned}$$

But the function $(x, y) \mapsto g_n(y) \cdot \mathbb{1}_Q(y) \cdot |f(x+y)|$ is Borel measurable, so that measurability of the integrated function $x \mapsto \int_{\mathbb{R}^d} g_n(y) \cdot \mathbb{1}_Q(y) \cdot |f(x+y)| dy$ follows from the Fubini-Tonelli theorem. Hence, $M_Q f$ is Borel measurable. \blacklozenge

It is easy to see that $\|\cdot\|_{W_Q(L_u^p)}$ satisfies the (quasi)-triangle inequality, since $M_Q(f+g) \leq M_Q f + M_Q g$. The remaining properties of a (quasi)-norm are also easy to check, possibly with the exception of definiteness. But this is a consequence of the following lemma:

Lemma 2.2. *For each Borel measurable unit neighborhood Q and each measurable $f : \mathbb{R}^d \rightarrow \mathbb{C}^k$, we have*

$$|f(x)| \leq (M_Q f)(x) \quad \text{for almost all } x \in \mathbb{R}^d.$$

In particular, $\|f\|_{L_u^p} \leq \|f\|_{W_Q(L_u^p)}$ and hence $f = 0$ almost everywhere if $\|f\|_{W_Q(L_u^p)} = 0$. \blacktriangleleft

Proof. Since Q is a unit neighborhood, there is $\varepsilon > 0$ with $B_{2\varepsilon}(0) \subset Q$. Since \mathbb{R}^d is second countable and since $(x + B_\varepsilon(0))_{x \in \mathbb{R}^d}$ is an open cover of \mathbb{R}^d , there is a countable family $(x_n)_{n \in \mathbb{N}}$ satisfying $\mathbb{R}^d = \bigcup_{n \in \mathbb{N}} (x_n + B_\varepsilon(0))$.

Now, for arbitrary $x \in \mathbb{R}^d$ and $y \in x + B_\varepsilon(0)$, we have $y + B_\varepsilon(0) \subset x + B_{2\varepsilon}(0) \subset x + Q$ and hence

$$(M_{B_\varepsilon(0)} f)(y) = \|f \cdot \mathbb{1}_{y+B_\varepsilon(0)}\|_{L^\infty} \leq \|f \cdot \mathbb{1}_{x+Q}\|_{L^\infty} = M_Q f(x) \quad \forall x \in \mathbb{R}^d \forall y \in x + B_\varepsilon(0).$$

Next, for each $n \in \mathbb{N}$, there is a null-set $N_n \subset x_n + B_\varepsilon(0)$ such that

$$|f(x)| \leq \|f \cdot \mathbb{1}_{x_n+B_\varepsilon(0)}\|_{L^\infty} = (M_{B_\varepsilon(0)} f)(x_n) \quad \text{for all } x \in [x_n + B_\varepsilon(0)] \setminus N_n.$$

But for each such x , there is some $\gamma \in B_\varepsilon(0)$ such that $x = x_n + \gamma$ and thus $x_n = x - \gamma \in x + B_\varepsilon(0)$, so that the equation from above yields $|f(x)| \leq (M_{B_\varepsilon(0)} f)(x_n) \leq M_Q f(x)$. Recall that this estimate holds for all $x \in [x_n + B_\varepsilon(0)] \setminus N_n$. Since $N := \bigcup_{n \in \mathbb{N}} N_n$ is a null-set and since $\mathbb{R}^d = \bigcup_{n \in \mathbb{N}} (x_n + B_\varepsilon(0))$, this completes the proof. \square

Although easy to prove, the following lemma is frequently helpful, since it shows that Schwartz functions are contained in arbitrary Wiener amalgam spaces.

Lemma 2.3. *For arbitrary $N \geq 0$, we have*

$$\left[M_{[-1,1]^d} (1 + |\bullet|)^{-N} \right] (x) \leq \left(1 + 2\sqrt{d} \right)^N \cdot (1 + |x|)^{-N} \quad \forall x \in \mathbb{R}^d.$$

In particular, if $p \in (0, \infty]$ is arbitrary and if we set $\|f\|_N := \sup_{x \in \mathbb{R}^d} (1 + |x|)^N \cdot |f(x)|$ for measurable $f : \mathbb{R}^d \rightarrow \mathbb{C}$, then

$$\|f\|_{W_{[-1,1]^d}(L_{(1+|\bullet|)^K}^p)} \leq \left(1 + 2\sqrt{d} \right)^N \cdot \left(\frac{1}{p} \frac{s_d}{N - K - \frac{d}{p}} \right)^{1/p} \cdot \|f\|_N \quad \text{as soon as} \quad N > K + \frac{d}{p}.$$

Hence, $\mathcal{S}(\mathbb{R}^d) \hookrightarrow W_{[-1,1]^d}(L_u^p)$ for all $p \in (0, \infty]$ and $u \in \left\{ v, v_0, (1 + |\bullet|)^K \right\}$. \blacktriangleleft

Proof. To prove the first claim, we distinguish two cases. For $|x| \leq 2\sqrt{d}$, note that $(1 + |x + a|)^{-N} \leq 1$ for all $a \in [-1, 1]^d$, so that we get

$$\left[M_{[-1,1]^d} (1 + |\bullet|)^{-N} \right] (x) \leq 1 \leq \left(1 + 2\sqrt{d} \right)^N \cdot (1 + |x|)^{-N}.$$

Otherwise, if $|x| \geq 2\sqrt{d}$, we have for $a \in [-1, 1]^d$ that

$$|x - a| \geq |x| - |a| \geq |x| - \sqrt{d} = \frac{|x|}{2} + \frac{|x|}{2} - \sqrt{d} \geq \frac{|x|}{2}$$

and hence $(1 + |x + a|)^{-N} \leq \left(1 + \frac{|x|}{2} \right)^{-N} \leq \left(\frac{1}{2} (1 + |x|) \right)^{-N} = 2^N (1 + |x|)^{-N}$. Since $2^N \leq \left(1 + 2\sqrt{d} \right)^N =: C$, we get all in all that $\left(M_{[-1,1]^d} (1 + |\bullet|)^{-N} \right) (x) \leq C \cdot (1 + |x|)^{-N}$ for all $x \in \mathbb{R}^d$, as claimed.

For the next claim, we can clearly assume $\|f\|_N < \infty$. In this case, we have $|f(x)| \leq \|f\|_N \cdot (1 + |x|)^{-N}$ for all $x \in \mathbb{R}^d$ and hence

$$\begin{aligned} (1 + |x|)^K \cdot \left(M_{[-1,1]^d} f \right) (x) &\leq \|f\|_N \cdot (1 + |x|)^K \cdot \left(M_{[-1,1]^d} (1 + |\bullet|)^{-N} \right) (x) \\ &\leq C \cdot \|f\|_N \cdot (1 + |x|)^{-(N-K)} \quad \forall x \in \mathbb{R}^d. \end{aligned}$$

This yields the claim, since equation (1.9) shows $\left\| (1 + |\bullet|)^{-(N-K)} \right\|_{L^p} \leq \left(\frac{1}{p} \frac{s_d}{N-K-\frac{d}{p}} \right)^{1/p} < \infty$, as long $N > K + \frac{d}{p}$.

The embedding $\mathcal{S}(\mathbb{R}^d) \hookrightarrow W_{[-1,1]^d}(L_u^p)$ for all $p \in (0, \infty]$ and $u \in \left\{ v, v_0, (1 + |\bullet|)^K \right\}$ is now trivial, since the norm $\|\bullet\|_N$ is continuous with respect to the topology on $\mathcal{S}(\mathbb{R}^d)$ and since we have

$$v(x) = v(0 + x) \leq v(0) \cdot v_0(x) \leq \Omega_1 v(0) \cdot (1 + |x|)^K \quad \forall x \in \mathbb{R}^d. \quad \square$$

Lemma 2.4. For $k \in \mathbb{N}$, a measurable $f : \mathbb{R}^d \rightarrow \mathbb{C}^k$, $T \in \text{GL}(\mathbb{R}^d)$ and a measurable $Q \subset \mathbb{R}^d$, we have

$$M_Q(f \circ T) = (M_{TQ}f) \circ T$$

and hence

$$\|f \circ T\|_{W_Q(L^p)} = |\det T|^{-1/p} \cdot \|f\|_{W_{TQ}(L^p)}. \quad \blacktriangleleft$$

Proof. Since T and T^{-1} map null-sets to null-sets, we have

$$\begin{aligned} [M_Q(f \circ T)](x) &= \text{ess sup}_{a \in Q} |(f \circ T)(x + a)| \\ &= \text{ess sup}_{b \in TQ} |f(b + Tx)| \\ &= (M_{TQ}f)(Tx) \end{aligned}$$

for all $x \in \mathbb{R}^d$. The final identity is a consequence of the definitions and of $\|f \circ T\|_{L^p} = |\det T|^{-1/p} \cdot \|f\|_{L^p}$, which follows easily from the change-of-variables formula. \square

Next, we show that iterated applications of M_Q can be estimated using a single $M_{Q'}$.

Lemma 2.5. Let $k \in \mathbb{N}$ and assume that $f : \mathbb{R}^d \rightarrow \mathbb{C}^k$ and $Q_1, Q_2 \subset \mathbb{R}^d$ are measurable and that $Q_1 + Q_2$ is also measurable. Then

$$M_{Q_1}[M_{Q_2}f] \leq M_{Q_1+Q_2}f.$$

In particular, for any measurable $u : \mathbb{R}^d \rightarrow (0, \infty)$, we have

$$\|M_{Q_2}f\|_{W_{Q_1}(L_u^p)} \leq \|f\|_{W_{Q_1+Q_2}^k(L_u^p)}. \quad \blacktriangleleft$$

Proof. For $a \in Q_1$, we have $a + Q_2 \subset Q_1 + Q_2$ and hence

$$(M_{Q_2}f)(x + a) = \|f \cdot \mathbf{1}_{x+a+Q_2}\|_{L^\infty} \leq \|f \cdot \mathbf{1}_{x+Q_1+Q_2}\|_{L^\infty} = (M_{Q_1+Q_2}f)(x).$$

Since this holds for all $a \in Q_1$, we get $(M_{Q_1}[M_{Q_2}f])(x) = \text{ess sup}_{a \in Q_1} (M_{Q_2}f)(x + a) \leq (M_{Q_1+Q_2}f)(x)$, as claimed. \square

The next three lemmas are important for us, since they imply $\|f\|_{W_{T_j^{-T}[-1,1]^d}(L_v^p)} \leq C_{i,j,p,v} \cdot \|f\|_{W_{T_i^{-T}[-1,1]^d}(L_v^p)}$, where the constant $C_{i,j,p,v}$ is explicitly known, cf. Corollary 2.9. We begin with an estimate for the norm of the translation operators on $L_v^p(\mathbb{R}^d)$.

Lemma 2.6. *For each $y \in \mathbb{R}^d$, the left-translation operator $L_y : L_v^p(\mathbb{R}^d) \rightarrow L_v^p(\mathbb{R}^d)$ is well-defined and bounded with*

$$\|L_y\| \leq v_0(y) \leq \Omega_1 \cdot (1 + |y|)^K. \quad \blacktriangleleft$$

Remark. The only property of v which is used in the proof is that $v(x+y) \leq v(x)v_0(y)$. By submultiplicativity of v_0 , the same estimate holds for v_0 instead of v , so that the claim of the lemma also holds for v_0 instead of v . \blacklozenge

Proof. Let $f \in L_v^p(\mathbb{R}^d)$ be arbitrary and note

$$\begin{aligned} v(x) \cdot |(L_y f)(x)| &= v((x-y)+y) \cdot |f(x-y)| \\ &\leq v_0(y) \cdot |(v \cdot f)(x-y)| \\ &= v_0(y) \cdot [L_y(v \cdot f)](x). \end{aligned}$$

By solidity and translation invariance of $L^p(\mathbb{R}^d)$, this implies

$$\|L_y f\|_{L_v^p} = \|v \cdot L_y f\|_{L^p} \leq v_0(y) \cdot \|L_y(v \cdot f)\|_{L^p} = v_0(y) \cdot \|v \cdot f\|_{L^p} = v_0(y) \cdot \|f\|_{L_v^p} < \infty. \quad \square$$

Now we can derive a first estimate which will allow us to switch from one “base set” Q to another one.

Lemma 2.7. *Let $Q_1, Q_2 \subset \mathbb{R}^d$ and assume that there are $x_1, \dots, x_N \in \mathbb{R}^d$ such that $Q_1 \subset \bigcup_{i=1}^N (x_i + Q_2)$. Let $p \in (0, \infty]$ and set $s := \min\{1, p\}$. Then we have*

$$\|f\|_{W_{Q_1}^k(L_v^p)} \leq \left(\sum_{i=1}^N [v_0(-x_i)]^s \right)^{1/s} \cdot \|f\|_{W_{Q_2}^k(L_v^p)} \leq \Omega_1 \cdot \left(\sum_{i=1}^N (1 + |x_i|)^{sK} \right)^{1/s} \cdot \|f\|_{W_{Q_2}^k(L_v^p)}.$$

for all measurable $f : \mathbb{R}^d \rightarrow \mathbb{C}^k$. \blacktriangleleft

Remark. • As for the previous lemma, the statement of the lemma remains true for v_0 instead of v .

• Note that if $Q_1, Q_2 \subset \mathbb{R}^d$ are two (Borel measurable) bounded unit-neighborhoods, compactness of $\overline{Q_1}$ yields finitely many $x_1, \dots, x_N \in \mathbb{R}^d$ satisfying $Q_1 \subset \overline{Q_1} \subset \bigcup_{i=1}^N (x_i + Q_2^c) \subset \bigcup_{i=1}^N (x_i + Q_2)$, so that the preceding lemma yields $\|f\|_{W_{Q_1}^k(L_v^p)} \lesssim \|f\|_{W_{Q_2}^k(L_v^p)}$, where the implied constant is independent of f . By symmetry, this argument shows $W_{Q_1}^k(L_v^p) = W_{Q_2}^k(L_v^p)$, with equivalent (quasi)-norms. But since the constants of the (quasi)-norm equivalence depend heavily on Q_1, Q_2 , this statement is not of too much value for us. \blacklozenge

Proof. We have for any measurable $f : \mathbb{R}^d \rightarrow \mathbb{C}^k$ that

$$\begin{aligned} (M_{Q_1} f)(x) &= \|f \cdot \mathbf{1}_{x+Q_1}\|_{L^\infty} \leq \left\| f \cdot \mathbf{1}_{x+\bigcup_{i=1}^N (x_i+Q_2)} \right\|_{L^\infty} \\ &\leq \sum_{i=1}^N \|f \cdot \mathbf{1}_{x+x_i+Q_2}\|_{L^\infty} \\ &= \sum_{i=1}^N (M_{Q_2} f)(x+x_i) \\ &= \sum_{i=1}^N (L_{-x_i} [M_{Q_2} f])(x). \end{aligned}$$

For $p \geq 1$, we can thus use the triangle inequality for L^p and the estimate for $\|L_y\|$ from Lemma 2.6, as well as solidity of L^p to derive

$$\|f\|_{W_{Q_1}^k(L_v^p)} = \|M_{Q_1} f\|_{L_v^p} \leq \sum_{i=1}^N \|L_{-x_i} (M_{Q_2} f)\|_{L_v^p} \leq \left[\sum_{i=1}^N v_0(-x_i) \right] \cdot \|M_{Q_2} f\|_{L_v^p} = \left[\sum_{i=1}^N v_0(-x_i) \right] \cdot \|f\|_{W_{Q_2}^k(L_v^p)}.$$

Similarly, for $p \in (0, 1)$, we use the p -triangle inequality (i.e., $\|\sum_{i=1}^n f_i\|_{L^p}^p \leq \sum_{i=1}^n \|f_i\|_{L^p}^p$) to derive

$$\|f\|_{W_{Q_1}^k(L_v^p)}^p = \|M_{Q_1} f\|_{L_v^p}^p \leq \sum_{i=1}^N \|L_{-x_i} (M_{Q_2} f)\|_{L_v^p}^p \leq \left[\sum_{i=1}^N (v_0(-x_i))^p \right] \cdot \|M_{Q_2} f\|_{L_v^p}^p = \left[\sum_{i=1}^N (v_0(-x_i))^p \right] \cdot \|f\|_{W_{Q_2}^k(L_v^p)}^p. \quad \square$$

In view of the preceding lemma, our next result becomes relevant:

Lemma 2.8. *Let $\|\cdot\|$ be any norm on \mathbb{R}^d and let $R > 0$. For any $r > 0$ and $N := \lfloor (3+2r)^d \rfloor$, there are $x_1, \dots, x_N \in B_{(1+r)R}^{\|\cdot\|}(0)$ satisfying*

$$B_{(1+r)R}^{\|\cdot\|}(0) \subset \bigcup_{i=1}^N \left[x_i + B_R^{\|\cdot\|}(0) \right],$$

where $B_s^{\|\cdot\|}(0) = \{x \in \mathbb{R}^d \mid \|x\| < s\}$. ◀

Proof. First of all, assume we are given $x_1, \dots, x_M \in B_{(1+r)R}^{\|\cdot\|}(0)$ such that $(x_i + B_{R/2}^{\|\cdot\|}(0))_{i \in \underline{M}}$ is pairwise disjoint. Because of $x_i \in B_{(1+r)R}^{\|\cdot\|}(0)$, we have $x_i + B_{R/2}^{\|\cdot\|}(0) \subset B_{(\frac{3}{2}+r)R}^{\|\cdot\|}(0)$, so that additivity and translation invariance of the Lebesgue-measure yields

$$\begin{aligned} M \cdot (R/2)^d \cdot \lambda(B_1^{\|\cdot\|}(0)) &= \sum_{i=1}^M \lambda(x_i + B_{R/2}^{\|\cdot\|}(0)) \\ &= \lambda\left(\biguplus_{i=1}^M x_i + B_{R/2}^{\|\cdot\|}(0)\right) \\ &\leq \lambda\left(B_{(\frac{3}{2}+r)R}^{\|\cdot\|}(0)\right) \\ &= \left[\left(\frac{3}{2} + r\right) R\right]^d \cdot \lambda(B_1^{\|\cdot\|}(0)) \end{aligned}$$

and thus $M \leq (3+2r)^d$. Since $M \in \mathbb{N}$, we even get $M \leq N$. In particular, there can be at most a finite number of such x_i .

Now (e.g. using Zorn's Lemma), we can find a *maximal* family $(x_i)_{i \in \underline{M}}$ in $B_{(1+r)R}^{\|\cdot\|}(0)$ such that the family of sets $(x_i + B_{R/2}^{\|\cdot\|}(0))_{i \in \underline{M}}$ is pairwise disjoint. As seen above, $M \leq N$.

It remains to show $B_{(1+r)R}^{\|\cdot\|}(0) \subset \bigcup_{i=1}^M [x_i + B_{R/2}^{\|\cdot\|}(0)] =: \Gamma$. Thus, let $x \in B_{(1+r)R}^{\|\cdot\|}(0)$ be arbitrary. In case of $x \in \{x_1, \dots, x_M\}$, we clearly have $x \in \Gamma$. But for $x \notin \{x_1, \dots, x_M\}$, we see by maximality of the family $(x_i)_{i \in \underline{M}}$ that there is some $i \in \underline{M}$ satisfying $(x + B_{R/2}^{\|\cdot\|}(0)) \cap (x_i + B_{R/2}^{\|\cdot\|}(0)) \neq \emptyset$. But this easily yields $x \in x_i + B_{R/2}^{\|\cdot\|}(0) \subset \Gamma$, as desired. ◻

As announced above, we can now derive a completely quantitative version of the (quasi)-norm equivalence between $W_{T_j^{-T}[-R,R]^d}^k(L_v^p)$ and $W_{T_i^{-T}[-L,L]^d}^k(L_v^p)$.

Corollary 2.9. *Let $i, j \in I$ and $p \in (0, \infty]$, let $R, L \in [1, \infty)$ and let $f : \mathbb{R}^d \rightarrow \mathbb{C}^k$ be measurable. Then we have*

$$\|f\|_{W_{T_j^{-T}[-R,R]^d}^k(L_v^p)} \leq \Omega_0^K \Omega_1 \cdot (3d(L+R))^{K+d \cdot \max\{1, \frac{1}{p}\}} \cdot (1 + \|T_j^{-1}T_i\|)^{K+d \cdot \max\{1, \frac{1}{p}\}} \cdot \|f\|_{W_{T_i^{-T}[-L,L]^d}^k(L_v^p)}. \quad \blacktriangleleft$$

Remark. As for the preceding results, the statement of the corollary remains valid if v is replaced by v_0 .

Finally, we explicitly state the two most important special cases of the preceding corollary:

- We have $i = j$ and $R = 2$, as well as $L = 1$. In this case, the corollary yields

$$\|f\|_{W_{T_i^{-T}[-2,2]^d}^k(L_v^p)} \leq \Omega_0^K \Omega_1 \cdot (18d)^{K+d \cdot \max\{1, \frac{1}{p}\}} \cdot \|f\|_{W_{T_i^{-T}[-1,1]^d}^k(L_v^p)}. \quad (2.2)$$

- We have $R = L = 1$. In this case, the corollary yields

$$\|f\|_{W_{T_j^{-T}[-1,1]^d}^k(L_v^p)} \leq \Omega_0^K \Omega_1 \cdot (6d)^{K+d \cdot \max\{1, \frac{1}{p}\}} \cdot (1 + \|T_j^{-1}T_i\|)^{K+d \cdot \max\{1, \frac{1}{p}\}} \cdot \|f\|_{W_{T_i^{-T}[-1,1]^d}^k(L_v^p)}. \quad (2.3) \quad \blacklozenge$$

Proof. For brevity, set $R' := \|(T_i^{-T})^{-1}T_j^{-T}\|_{\ell^\infty \rightarrow \ell^\infty} \cdot R = \|T_j^{-1}T_i\|_{\ell^1 \rightarrow \ell^1} \cdot R$ and note

$$(T_i^{-T})^{-1}T_j^{-T}[-R,R]^d \subset [-R', R']^d = \overline{B_{R'}^{\|\cdot\|_\infty}(0)} \subset \overline{B_{(1+\frac{R'}{L})}^{\|\cdot\|_\infty}(0)}.$$

But Lemma 2.8 yields certain $x_1, \dots, x_N \in \overline{B_{\left(1+\frac{R'}{L}\right)L}^{\|\cdot\|_\infty}}(0)$ satisfying

$$\overline{B_{\left(1+\frac{R'}{L}\right)L}^{\|\cdot\|_\infty}}(0) \subset \bigcup_{i=1}^N \left(x_i + \overline{B_L^{\|\cdot\|_\infty}}(0) \right) = \bigcup_{i=1}^N \left(x_i + [-L, L]^d \right),$$

where

$$N \leq \left(3 + 2\frac{R'}{L} \right)^d = \left(3 + 2\frac{R}{L} \|T_j^{-1}T_i\|_{\ell^1 \rightarrow \ell^1} \right)^d \leq 3^d \left(1 + \frac{R}{L} \right)^d \left(1 + \|T_j^{-1}T_i\|_{\ell^1 \rightarrow \ell^1} \right)^d.$$

Next, note $\|T_j^{-1}T_i\|_{\ell^1 \rightarrow \ell^1} \leq \sqrt{d} \cdot \|T_j^{-1}T_i\|$ and hence $N \leq \left(3\sqrt{d} \right)^d \left(1 + \frac{R}{L} \right)^d \left(1 + \|T_j^{-1}T_i\| \right)^d$.

By putting together what we derived above, we get

$$T_j^{-T}[-R, R]^d \subset \bigcup_{i=1}^N \left(T_i^{-T}x_i + T_i^{-T}[-L, L]^d \right).$$

Next, we set $s := \min\{1, p\}$, note

$$\begin{aligned} \|x_i\|_\infty &\leq \left(1 + \frac{R'}{L} \right) L = (L + R') \\ &= \left(L + R \cdot \|T_j^{-1}T_i\|_{\ell^1 \rightarrow \ell^1} \right) \\ &\leq (L + R) \left(1 + \|T_j^{-1}T_i\|_{\ell^1 \rightarrow \ell^1} \right) \\ &\leq \sqrt{d} \cdot (L + R) \left(1 + \|T_j^{-1}T_i\| \right) \end{aligned}$$

and thus $1 + |x_i| \leq 1 + d \cdot (L + R) \left(1 + \|T_j^{-1}T_i\| \right) \leq d \cdot (1 + L + R) \left(1 + \|T_j^{-1}T_i\| \right)$ and recall equation (1.11) to derive

$$\begin{aligned} \left[\sum_{i=1}^N \left(1 + |T_i^{-T}x_i| \right)^{sK} \right]^{1/s} &\leq \Omega_0^K \cdot \left[\sum_{i=1}^N \left(1 + |x_i| \right)^{sK} \right]^{1/s} \\ &\leq [d\Omega_0(1 + L + R)(1 + \|T_j^{-1}T_i\|)]^K \cdot N^{1/s} \\ (\text{since } R, L \geq 1) &\leq 3^{K+\frac{d}{s}} d^{K+\frac{d}{2s}} (L + R)^{K+\frac{d}{s}} \cdot \Omega_0^K \cdot \left(1 + \|T_j^{-1}T_i\| \right)^{K+\frac{d}{s}} \\ &\leq \Omega_0^K \cdot (3d(L + R))^{K+\frac{d}{s}} \cdot \left(1 + \|T_j^{-1}T_i\| \right)^{K+\frac{d}{s}}. \end{aligned}$$

All in all, Lemma 2.7 implies

$$\|f\|_{W_{T_j^{-T}[-R, R]^d}^k(L_v^p)} \leq \Omega_0^K \Omega_1 \cdot (3d(L + R))^{K+\frac{d}{s}} \cdot \left(1 + \|T_j^{-1}T_i\| \right)^{K+\frac{d}{s}} \cdot \|f\|_{W_{T_i^{-T}[-L, L]^d}^k(L_v^p)},$$

as desired. \square

2.2. The oscillation of a function. Later in the paper, we will need to discretize certain reproducing formulas involving convolutions. As observed in [25, 26, 27, 44, 2], a central tool for these discretizations is the oscillation of a function and certain properties of and estimates for it. The goal of this subsection is to collect these properties and estimates.

Definition 2.10. Let $f : \mathbb{R}^d \rightarrow \mathbb{C}^k$ and let $\emptyset \neq Q \subset \mathbb{R}^d$. We define the Q -**oscillation** of f by

$$\text{osc}_Q f : \mathbb{R}^d \rightarrow [0, \infty], x \mapsto \sup_{y, z \in x+Q} |f(y) - f(z)| = \sup_{a, b \in Q} |f(x+a) - f(x+b)|. \quad \blacktriangleleft$$

Remark. Note that if f is continuous, then so is $x \mapsto |f(x+a) - f(x+b)|$, so that $\text{osc}_Q f$ is lower semicontinuous and hence measurable. \blacklozenge

As our first step, we investigate some elementary properties of the oscillation, in particular the behaviour of the oscillation under a linear change of variables and under convolution.

Lemma 2.11. Let $f : \mathbb{R}^d \rightarrow \mathbb{C}^k$, $T \in \text{GL}(\mathbb{R}^d)$ and let $\emptyset \neq Q \subset \mathbb{R}^d$. Then

$$\text{osc}_Q(f \circ T) = (\text{osc}_{TQ} f) \circ T. \quad \blacktriangleleft$$

Proof. We have

$$\begin{aligned} [\operatorname{osc}_Q (f \circ T)](x) &= \sup_{a, b \in Q} |(f \circ T)(x + a) - (f \circ T)(x + b)| \\ &= \sup_{\alpha, \beta \in TQ} |f(\alpha + Tx) - f(\beta + Tx)| \\ &= (\operatorname{osc}_{TQ} f)(Tx). \end{aligned} \quad \square$$

Lemma 2.12. *Let $f, g : \mathbb{R}^d \rightarrow \mathbb{C}$ be measurable, let $\emptyset \neq Q \subset \mathbb{R}^d$ and assume that $\operatorname{osc}_Q f$ is measurable and that $(|f| * |g|)(x) < \infty$ for every $x \in \mathbb{R}^d$. Then*

$$[\operatorname{osc}_Q (f * g)](x) \leq [(\operatorname{osc}_Q f) * |g|](x) \quad \forall x \in \mathbb{R}^d. \quad \blacktriangleleft$$

Proof. Let $x \in \mathbb{R}^d$ and fix $a, b \in Q$. Then

$$\begin{aligned} |(f * g)(x + a) - (f * g)(x + b)| &= \left| \int_{\mathbb{R}^d} f((x + a) - y) g(y) \, dy - \int_{\mathbb{R}^d} f((x + b) - y) g(y) \, dy \right| \\ &\leq \int_{\mathbb{R}^d} |f((x - y) + a) - f((x - y) + b)| \cdot |g(y)| \, dy \\ &\leq \int_{\mathbb{R}^d} (\operatorname{osc}_Q f)(x - y) \cdot |g(y)| \, dy \\ &= [(\operatorname{osc}_Q f) * |g|](x), \end{aligned}$$

as claimed. \square

Intuitively, it should be true that smooth functions have a small oscillation if their derivative is small. The next two lemmas make this precise:

Lemma 2.13. *Let $f \in C^1(\mathbb{R}^d; \mathbb{C})$. Then we have for every bounded, convex set $Q \subset \mathbb{R}^d$ with nonempty interior that*

$$\operatorname{osc}_Q f \leq \operatorname{diam}(Q) \cdot M_Q(\nabla f). \quad \blacktriangleleft$$

Proof. For $x \in \mathbb{R}^d$ and $a, b \in Q$, the fundamental theorem of calculus yields

$$\begin{aligned} |f(x + b) - f(x + a)| &= \left| \int_0^1 \frac{d}{ds} f(x + a + s(b - a)) \, ds \right| \\ &= \left| \int_0^1 \langle \nabla f(x + a + s(b - a)), b - a \rangle \, ds \right| \\ &\leq \operatorname{diam}(Q) \cdot \sup_{s \in [0, 1]} |\nabla f(x + sb + (1 - s)a)| \\ (Q \text{ convex}) &\leq \operatorname{diam}(Q) \cdot \sup_{c \in Q} |\nabla f(x + c)| \\ &\stackrel{(\dagger)}{\leq} \operatorname{diam}(Q) \cdot [M_Q(\nabla f)](x). \end{aligned}$$

Here, it only remains to justify the last step, i.e. that $[M_Q(\nabla f)](x) = \operatorname{ess\,sup}_{c \in Q} |\nabla f(x + c)| \stackrel{!}{=} \sup_{c \in Q} |\nabla f(x + c)|$. Here, only “ \geq ” is nontrivial. But by continuity of ∇f , and since nonempty open sets have positive measure, it is not hard to see

$$\operatorname{ess\,sup}_{c \in Q} |\nabla f(x + c)| \geq \sup_{c \in Q^\circ} |\nabla f(x + c)|,$$

so that it suffices (by continuity) to show that $\overline{Q^\circ} \supset Q$. But for arbitrary $a \in Q^\circ$ and $b \in Q$, we have $B_\varepsilon(a) \subset Q$ for some $\varepsilon > 0$. For $t \in (0, 1)$, this implies

$$ta + (1 - t)b \in t \cdot B_\varepsilon(a) + (1 - t)b \subset Q$$

and hence $ta + (1 - t)b \in Q^\circ$. Because of $ta + (1 - t)b \xrightarrow{t \rightarrow 0} b$, we conclude $b \in \overline{Q^\circ}$, as desired. \square

Lemma 2.14. *Let $f \in C^1(\mathbb{R}^d; \mathbb{C})$ and $N \geq 0$ and set $C := (3\sqrt{d})^{N+1}$. Then*

$$(\operatorname{osc}_{\delta \cdot [-1, 1]^d} f)(x) \leq C \cdot \|\nabla f\|_N \cdot \delta \cdot (1 + |x|)^{-N} \quad \forall x \in \mathbb{R}^d \, \forall \delta \in (0, 1], \quad (2.4)$$

where $\|\nabla f\|_N := \sup_{x \in \mathbb{R}^d} (1 + |x|)^N |\nabla f(x)|$. \blacktriangleleft

Proof. By Cauchy-Schwarz, we have $\text{diam}(\delta \cdot [-1, 1]^d) \leq 2\sqrt{d} \cdot \delta$. We can clearly assume $\|\nabla f\|_N < \infty$. Since we also have $\delta[-1, 1]^d \subset [-1, 1]^d$, we get from Lemmas 2.13 and 2.3 that

$$\begin{aligned} (\text{osc}_{\delta[-1, 1]^d} f)(x) &\leq 2\sqrt{d} \cdot \delta \cdot \left[M_{\delta[-1, 1]^d}(\nabla f) \right](x) \\ &\leq 2\sqrt{d} \cdot \delta \cdot \left[M_{[-1, 1]^d}(\nabla f) \right](x) \\ &\leq 2\sqrt{d} \cdot \delta \cdot \|\nabla f\|_N \cdot \left[M_{[-1, 1]^d}(1 + |\bullet|)^{-N} \right](x) \\ &\stackrel{(\text{Lemma 2.3})}{\leq} 2\sqrt{d} \cdot \left(1 + 2\sqrt{d}\right)^N \cdot \delta \cdot \|\nabla f\|_N \cdot (1 + |x|)^{-N} \\ &\stackrel{(\text{since } 1 \leq \sqrt{d})}{\leq} \left(3\sqrt{d}\right)^{N+1} \cdot \delta \cdot \|\nabla f\|_N \cdot (1 + |x|)^{-N} \end{aligned}$$

for all $x \in \mathbb{R}^d$. □

2.3. Self-improving properties for bandlimited functions. Our next aim is to show that bandlimited L^p -functions automatically belong to $W_Q(L^p)$. More precisely, we will show for $\text{supp } \hat{f} \subset T_i[-R, R]^d + \xi_0$ that

$$\|f\|_{W_{T_i^{-T}[-1, 1]^d}(L^p)} \leq C_{d,p,R} \cdot \|f\|_{L^p},$$

which we call a “self-improving property”, since we can improve a simple L^p estimate to a Wiener-amalgam estimate, at least for suitably bandlimited functions. In fact, we will develop a *weighted version* of the preceding estimate.

All of our results in this section are based on the following convolution relation for bandlimited L^p -functions, which we take from [77, Theorem 3.4]. We remark that this *pointwise* estimate already appears in the proof of [73, Proposition 1.5.1], but is not stated explicitly as a theorem.

Theorem 2.15. Let $Q, \Omega \subset \mathbb{R}^d$ be compact and let $p \in (0, 1]$. Furthermore, let $\psi \in L^1(\mathbb{R}^d)$ with $\text{supp } \psi \subset Q$ and such that $\mathcal{F}^{-1}\psi \in L^p(\mathbb{R}^d)$.

For each $f \in L^p(\mathbb{R}^d) \cap \mathcal{S}'(\mathbb{R}^d)$ with $\text{supp } \hat{f} \subset \Omega$, we have $\mathcal{F}^{-1}(\psi \cdot \hat{f}) = (\mathcal{F}^{-1}\psi) * f \in L^p(\mathbb{R}^d)$ with

$$(|\mathcal{F}^{-1}\psi| * |f|)(x) \leq [\lambda_d(Q - \Omega)]^{\frac{1}{p}-1} \cdot \left[\int_{\mathbb{R}^d} |(\mathcal{F}^{-1}\psi)(y)|^p \cdot |f(x-y)|^p dy \right]^{1/p}$$

for all $x \in \mathbb{R}^d$. ◀

In order to “circumvent” the assumption $f \in L^p(\mathbb{R}^d)$, we also need the following approximation result, a proof of which can be found in [77, Lemma 3.2], or in [73, Theorem 1.4.1]. In fact, the proof given in [77] is based on that in [73].

Lemma 2.16. Let $\Omega \subset \mathbb{R}^d$ be compact and assume $f \in \mathcal{S}'(\mathbb{R}^d)$ with $\text{supp } \hat{f} \subset \Omega$. Then f is given by (integration against) a smooth function $g \in C^\infty(\mathbb{R}^d)$ with polynomially bounded derivatives of all orders.

Furthermore, there is a sequence of Schwartz functions $(g_n)_{n \in \mathbb{N}}$ with the following properties:

- (1) $|g_n(x)| \leq |g(x)|$ for all $x \in \mathbb{R}^d$,
- (2) $g_n(x) \xrightarrow{n \rightarrow \infty} g(x)$ for all $x \in \mathbb{R}^d$,
- (3) $\text{supp } \hat{g}_n \subset B_{1/n}(\Omega)$, where $B_{1/n}(\Omega)$ is the $\frac{1}{n}$ -neighborhood of Ω , given by

$$B_{1/n}(\Omega) = \left\{ \xi \in \mathbb{R}^d \mid \text{dist}(\xi, \Omega) < \frac{1}{n} \right\}.$$

In the following, we will identify the bandlimited distribution f with its “smooth version” g . ◀

Using the two preceding results, we can now establish our first “self-improving property”.

Theorem 2.17. For $p \in (0, \infty]$, $\xi_0 \in \mathbb{R}^d$ and $f \in \mathcal{S}'(\mathbb{R}^d)$ with $\text{supp } \hat{f} \subset T_i[-R, R]^d + \xi_0$ for some $i \in I$, we have

$$\|f\|_{W_{T_i^{-T}[-1, 1]^d}(L_v^p)} \leq C \cdot \|f\|_{L_v^p}$$

with $s := \min\{1, p\}$ and $N := \lceil K + \frac{d+1}{s} \rceil$, as well as

$$C = 2^{4(1+\frac{d}{s})} s^{\frac{1}{s}} \left(192 \cdot d^{3/2} \cdot N \right)^{N+1} \cdot \Omega_0^K \Omega_1 \cdot (1+R)^{\frac{d}{s}}. \quad \blacktriangleleft$$

Remark. The only specific property of v which is used in the proof is that $v(x+y) \leq v(x)v_0(y)$. By submultiplicativity of v_0 , the same remains true when v is replaced by v_0 . Hence, we also have $\|f\|_{W_{T_i^{-T}[-1,1]^d}(L_{v_0}^p)} \leq C \cdot \|f\|_{L_{v_0}^p}$

for $f \in \mathcal{S}'(\mathbb{R}^d)$ with $\text{supp } \widehat{f} \subset T_i[-R, R]^d + \xi_0$. \blacklozenge

Proof. We can clearly assume $\|f\|_{L_v^p} < \infty$, since otherwise the claim is trivial. Using Lemma 2.16, choose a sequence of Schwartz functions $(f_n)_{n \in \mathbb{N}}$ satisfying $|f_n(x)| \leq |f(x)|$, as well as $f_n(x) \xrightarrow{n \rightarrow \infty} f(x)$ and furthermore $\text{supp } \widehat{f_n} \subset B_{1/n}(T_i[-R, R]^d + \xi_0)$ for all $n \in \mathbb{N}$. Note that $T_i(-(R+1), R+1)^d + \xi_0$ is a neighborhood of the compact(!) set $T_i[-R, R]^d + \xi_0$, so that there is some $n_0 \in \mathbb{N}$ satisfying $\text{supp } \widehat{f_n} \subset T_i(-(R+1), R+1)^d + \xi_0$ for all $n \geq n_0$. By dropping (or modifying) the first n_0 terms of the sequence $(f_n)_{n \in \mathbb{N}}$, we can assume that this holds for all $n \in \mathbb{N}$.

Let $s = \min\{1, p\}$ and $N = \lceil K + \frac{d+1}{s} \rceil \geq 2$ as in the statement of the theorem. Using Lemma A.2 (with $R+1$ instead of R and with $s = 3$) and Corollary A.3, we get a function $\psi \in C_c^\infty(\mathbb{R}^d)$ satisfying $0 \leq \psi \leq 1$, as well as $\text{supp } \psi \subset [-(R+4), R+4]^d$ and $\psi \equiv 1$ on $[-(R+1), R+1]^d$ which also satisfies

$$\begin{aligned} |(\mathcal{F}^{-1}\psi)(x)| &\leq 2\pi \cdot 2^d \cdot (48d)^{N+1} (N+2)! \cdot (R+4)^d \cdot (1+|x|)^{-N} \\ &=: C_1 \cdot (1+|x|)^{-N} \end{aligned}$$

for all $x \in \mathbb{R}^d$. Note because of $N \geq 2$ that $(N+2)! = \prod_{\ell=1}^{N+2} \ell \leq \prod_{\ell=2}^{N+2} (N+2) = (N+2)^{N+1} \leq (2N)^{N+1}$ and hence

$$C_1 \leq 2^{3(1+d)} (96d \cdot N)^{N+1} \cdot (1+R)^d.$$

Next, define

$$\varrho := \mathcal{F}^{-1}(L_{\xi_0}[\psi \circ T_i^{-1}]) = |\det T_i| \cdot M_{\xi_0}[(\mathcal{F}^{-1}\psi) \circ T_i^T]$$

and note $\widehat{\varrho} \equiv 1$ on $T_i[-(R+1), R+1]^d + \xi_0$, so that $\widehat{f_n} = \widehat{\varrho} \cdot \widehat{f_n}$, which implies $f_n = f_n * \varrho$. Next, we note for arbitrary $x \in \mathbb{R}^d$ and $y \in T_i^{-T}[-1, 1]^d$ because of

$$\begin{aligned} 1 + |T_i^T x| &\leq 1 + |T_i^T(x+y)| + |-T_i^T y| \\ &\leq 1 + \sqrt{d} + |T_i^T(x+y)| \\ &\leq (1 + \sqrt{d}) (1 + |T_i^T(x+y)|) \end{aligned}$$

that

$$\begin{aligned} |\varrho(x+y)| &= |\det T_i| \cdot |(\mathcal{F}^{-1}\psi)(T_i^T(x+y))| \\ &\leq C_1 \cdot |\det T_i| \cdot (1 + |T_i^T(x+y)|)^{-N} \\ &\leq (1 + \sqrt{d})^N C_1 \cdot |\det T_i| \cdot (1 + |T_i^T x|)^{-N}. \end{aligned} \tag{2.5}$$

Now, we distinguish the two cases $p \in [1, \infty]$ and $p \in (0, 1)$.

Case 1: We have $p \in [1, \infty]$ and hence $s = 1$. In this case, note because of

$$\begin{aligned} v(x) &= v(z+x-z) \leq v(z)v_0(x-z) \\ &\leq \Omega_1 \cdot v(z) (1+|x-z|)^K \\ (\text{eq. (1.11)}) &\leq \Omega_0^K \Omega_1 \cdot v(z) \cdot (1 + |T_i^T(x-z)|)^K \end{aligned} \tag{2.6}$$

that

$$\begin{aligned} v(x) \cdot \left(M_{T_i^{-T}[-1,1]^d} f_n \right)(x) &\leq v(x) \cdot \sup_{y \in T_i^{-T}[-1,1]^d} |f_n(x+y)| \\ (\text{since } f_n = f_n * \varrho) &\leq v(x) \cdot \sup_{y \in T_i^{-T}[-1,1]^d} \int_{\mathbb{R}^d} |\varrho((x-z)+y)| \cdot |f_n(z)| \, dz \\ (\text{eq. (2.5) with } x-z \text{ instead of } x) &\leq (1 + \sqrt{d})^N C_1 \cdot |\det T_i| \cdot \int_{\mathbb{R}^d} v(x) \cdot (1 + |T_i^T(x-z)|)^{-N} \cdot |f_n(z)| \, dz \\ (\text{eq. (2.6)}) &\leq (1 + \sqrt{d})^N \Omega_0^K \Omega_1 C_1 \cdot |\det T_i| \cdot \int_{\mathbb{R}^d} (1 + |T_i^T(x-z)|)^{K-N} \cdot |(v \cdot f_n)(z)| \, dz. \end{aligned}$$

Now, we simply use Young's inequality $\|f * g\|_{L^p} \leq \|f\|_{L^1} \cdot \|g\|_{L^p}$ to conclude

$$\begin{aligned} \|f_n\|_{W_{T_i^{-T}[-1,1]^d}(L_v^p)} &= \left\| v \cdot M_{T_i^{-T}[-1,1]^d} f_n \right\|_{L^p} \\ &\leq \left(1 + \sqrt{d}\right)^N \Omega_0^K \Omega_1 C_1 \cdot \left\| (1 + |T_i^T \bullet|)^{-(N-K)} \right\|_{L^1} \cdot \|v \cdot f_n\|_{L^p} \\ (N-K \geq d+1, \text{ eq. (1.9) and } |f_n| \leq |f|) &\leq \left(1 + \sqrt{d}\right)^N s_d \Omega_0^K \Omega_1 C_1 \cdot \|f\|_{L_v^p}. \end{aligned}$$

But because of $f_n \rightarrow f$ pointwise, we get

$$\left(M_{T_i^{-T}[-1,1]^d} f \right) (x) = \left\| \mathbf{1}_{x+T_i^{-T}[-1,1]^d} \cdot f \right\|_{L^\infty} \leq \liminf_{n \rightarrow \infty} \left\| \mathbf{1}_{x+T_i^{-T}[-1,1]^d} \cdot f_n \right\|_{L^\infty} = \liminf_{n \rightarrow \infty} \left(M_{T_i^{-T}[-1,1]^d} f_n \right) (x),$$

so that an application of Fatou's Lemma yields

$$\|f\|_{W_{T_i^{-T}[-1,1]^d}(L_v^p)} \leq \liminf_{n \rightarrow \infty} \|f_n\|_{W_{T_i^{-T}[-1,1]^d}(L_v^p)} \leq \left(1 + \sqrt{d}\right)^N s_d \Omega_0^K \Omega_1 C_1 \cdot \|f\|_{L_v^p},$$

as desired.

Case 2: We have $p \in (0, 1)$ and hence $s = p$. In this case, we first note

$$\begin{aligned} \lambda_d \left(\widehat{\text{supp } f_n} - \widehat{\text{supp } \varrho} \right) &\leq \lambda_d \left(\left[T_i[-(R+1), R+1]^d + \xi_0 \right] - \left[T_i[-(R+4), R+4]^d + \xi_0 \right] \right) \\ &\leq \lambda_d \left(T_i[-(2R+5), 2R+5]^d \right) \\ &= |\det T_i| \cdot (4R+10)^d. \end{aligned}$$

For brevity, set $C_2 := \left(1 + \sqrt{d}\right)^N (4R+10)^{d(\frac{1}{p}-1)} C_1$ and apply Theorem 2.15 to get

$$\begin{aligned} v(x) \cdot \left(M_{T_i^{-T}[-1,1]^d} f_n \right) (x) &\leq v(x) \cdot \sup_{y \in T_i^{-T}[-1,1]^d} |f_n(x+y)| \\ (\text{since } f_n = f_n * \varrho) &\leq v(x) \cdot \sup_{y \in T_i^{-T}[-1,1]^d} (|f_n| * |\varrho|)(x+y) \\ (\text{Theorem 2.15}) &\leq \left[(4R+10)^d \cdot |\det T_i| \right]^{\frac{1}{p}-1} \cdot v(x) \cdot \sup_{y \in T_i^{-T}[-1,1]^d} \left(\int_{\mathbb{R}^d} |\varrho((x-z)+y)|^p \cdot |f_n(z)|^p \, dz \right)^{1/p} \\ (\text{eq. (2.5)}) &\leq C_2 \cdot |\det T_i|^{1/p} \cdot v(x) \cdot \left(\int_{\mathbb{R}^d} (1 + |T_i^T(x-z)|)^{-Np} \cdot |f_n(z)|^p \, dz \right)^{1/p} \\ (\text{eq. (2.6)}) &\leq C_2 \Omega_0^K \Omega_1 \cdot |\det T_i|^{1/p} \cdot \left(\int_{\mathbb{R}^d} (1 + |T_i^T(x-z)|)^{-p(N-K)} \cdot |(v \cdot f_n)(z)|^p \, dz \right)^{1/p}. \end{aligned}$$

Finally, take the L^p -norm of the preceding estimate to conclude

$$\begin{aligned} \|f_n\|_{W_{T_i^{-T}[-1,1]^d}(L_v^p)}^p &\leq (C_2 \Omega_0^K \Omega_1)^p \cdot |\det T_i| \cdot \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (1 + |T_i^T(x-z)|)^{-p(N-K)} \cdot |(v \cdot f_n)(z)|^p \, dz \, dx \\ (\text{Fubini and } y=x-z) &= (C_2 \Omega_0^K \Omega_1)^p \cdot \|f\|_{L_v^p}^p \cdot |\det T_i| \cdot \int_{\mathbb{R}^d} (1 + |T_i^T y|)^{-p(N-K)} \, dy \\ &= (C_2 \Omega_0^K \Omega_1)^p \cdot \left\| (1 + |\bullet|)^{-(N-K)} \right\|_{L^p}^p \cdot \|f\|_{L_v^p}^p \\ (\text{eq. (1.9) and } N-K-\frac{d}{p} \geq \frac{1}{p}) &\leq (C_2 \Omega_0^K \Omega_1)^p \cdot s_d \cdot \|f\|_{L_v^p}^p. \end{aligned}$$

The remainder of the proof is now as for $p \in [1, \infty]$, but with a slightly different constant. \square

Now, we establish our second “self-improving property”, which yields an estimate for the L_v^p -norm of the oscillation of a band-limited function, only in terms of the L_v^p -norm of the function.

Theorem 2.18. For each $p \in [0, \infty)$, $i \in I$, $\xi_0 \in \mathbb{R}^d$, $R > 0$ and $f \in \mathcal{S}'(\mathbb{R}^d)$ with $\text{supp } \widehat{f} \subset T_i[-R, R]^d + \xi_0$, we have

$$\left\| \text{osc}_{\delta \cdot T_i^{-T}[-1,1]^d} [M_{-\xi_0} f] \right\|_{W_{T_i^{-T}[-1,1]^d}(L_v^p)} \leq C \cdot \delta \cdot \|f\|_{L_v^p}$$

with

$$C := \Omega_0^{2K} \Omega_1^2 \cdot \left(23040 \cdot d^{3/2} \cdot \left(K + 1 + \frac{d+1}{\min\{1, p\}} \right) \right)^{K+2+\frac{d+1}{\min\{1, p\}}} \cdot (1+R)^{1+\frac{d}{\min\{1, p\}}}. \quad \blacktriangleleft$$

Remark. As usual, the claim remains valid when v is replaced by v_0 throughout. \blacklozenge

Proof. As usual, since f is a *bandlimited* tempered distribution, it is actually given by integration against a smooth function with polynomially bounded derivatives. Furthermore, $\text{supp } \mathcal{F}[M_{-\xi_0} f] = \text{supp } L_{-\xi_0} \widehat{f} \subset T_i[-R, R]^d$, so that we can assume $\xi_0 = 0$ for the remainder of the proof.

The first part of the proof is now very similar to that of Theorem 2.17: We can clearly assume $\|f\|_{L_v^p} < \infty$, since otherwise the claim is trivial. Using Lemma 2.16, choose a sequence of Schwartz functions $(f_n)_{n \in \mathbb{N}}$ satisfying $|f_n(x)| \leq |f(x)|$, as well as $f_n(x) \xrightarrow{n \rightarrow \infty} f(x)$ and furthermore $\text{supp } \widehat{f_n} \subset B_{1/n}(T_i[-R, R]^d)$ for all $n \in \mathbb{N}$. Note that $T_i(-(R+1), R+1)^d$ is a neighborhood of the compact(!) set $T_i[-R, R]^d$, so that there is some $n_0 \in \mathbb{N}$ satisfying $\text{supp } \widehat{f_n} \subset T_i[-(R+1), R+1]^d$ for all $n \geq n_0$. By dropping (or modifying) the first n_0 terms of the sequence $(f_n)_{n \in \mathbb{N}}$, we can assume that this holds for all $n \in \mathbb{N}$.

Let $s := \min\{1, p\}$ and $N := \lceil K + \frac{d+1}{s} \rceil \geq 2$. Using Lemma A.2 (with $R+1$ instead of R and with $s = 3$) and Corollary A.3, we get a function $\psi \in C_c^\infty(\mathbb{R}^d)$ satisfying $0 \leq \psi \leq 1$, as well as $\text{supp } \psi \subset [-(R+4), R+4]^d$ and $\psi \equiv 1$ on $[-(R+1), R+1]^d$ which also satisfies

$$\begin{aligned} |(\partial^\alpha [\mathcal{F}^{-1}\psi])(x)| &\leq 2\pi \cdot 2^d \cdot (48d)^{N+1} (N+2)! \cdot (R+5)^{|\alpha|} (R+4)^d \cdot (1+|x|)^{-N} \\ &\leq 2\pi \cdot 2^d \cdot (48d)^{N+1} (N+2)! \cdot (R+5)^{d+1} \cdot (1+|x|)^{-N} \\ &=: C_1 \cdot (1+|x|)^{-N} \end{aligned} \quad (2.7)$$

for all $x \in \mathbb{R}^d$ and $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq 1$. Using $(N+2)! = \prod_{\ell=1}^{N+2} \ell \leq \prod_{\ell=2}^{N+2} (N+2) = (N+2)^{N+1} \leq (2N)^{N+1}$, we get

$$C_1 \leq 40 \cdot 10^d \cdot (96d \cdot N)^{N+1} \cdot (1+R)^{d+1}. \quad (2.8)$$

Now, let $g_n := f_n \circ T_i^{-T}$ for $n \in \mathbb{N}$. Using Lemmas 2.11 and 2.13, we see

$$\begin{aligned} \text{osc}_{\delta \cdot T_i^{-T}[-1,1]^d} f_n &= \text{osc}_{\delta \cdot T_i^{-T}[-1,1]^d} (g_n \circ T_i^T) \\ &\stackrel{\text{(Lemma 2.11)}}{=} \left(\text{osc}_{\delta \cdot [-1,1]^d} g_n \right) \circ T_i^T \\ &\stackrel{\text{(Lemma 2.13)}}{\leq} 2\sqrt{d} \cdot \delta \cdot \left(M_{\delta[-1,1]^d} [\nabla g_n] \right) \circ T_i^T. \end{aligned}$$

Based on this estimate, Lemmas 2.4 and 2.5 show

$$\begin{aligned} M_{T_i^{-T}[-1,1]^d} \left[\text{osc}_{\delta \cdot T_i^{-T}[-1,1]^d} f_n \right] &\leq 2\sqrt{d} \cdot \delta \cdot M_{T_i^{-T}[-1,1]^d} \left[\left(M_{\delta[-1,1]^d} [\nabla g_n] \right) \circ T_i^T \right] \\ &\stackrel{\text{(Lemma 2.4)}}{=} 2\sqrt{d} \cdot \delta \cdot \left[M_{[-1,1]^d} \left(M_{\delta[-1,1]^d} [\nabla g_n] \right) \right] \circ T_i^T \\ &\stackrel{\text{(Lemma 2.5 and } \delta \leq 1)}{\leq} 2\sqrt{d} \cdot \delta \cdot \left(M_{[-2,2]^d} [\nabla g_n] \right) \circ T_i^T \\ &\stackrel{\text{(Lemma 2.4)}}{=} 2\sqrt{d} \cdot \delta \cdot M_{T_i^{-T}[-2,2]^d} \left[(\nabla g_n) \circ T_i^T \right]. \end{aligned} \quad (2.9)$$

Next, observe $\text{supp } \widehat{g_n} = \text{supp } [|\det T_i| \cdot \widehat{f_n} \circ T_i] = T_i^{-1} \text{supp } \widehat{f_n} \subset [-(R+1), R+1]^d$ for all $n \in \mathbb{N}$, so that we see $\widehat{g_n} = \widehat{g_n} \cdot \psi$. Hence, $g_n = g_n * \mathcal{F}^{-1}\psi$. Because of $g_n, \mathcal{F}^{-1}\psi \in \mathcal{S}(\mathbb{R}^d)$, this easily implies $\nabla g_n = g_n * \nabla(\mathcal{F}^{-1}\psi)$. But for arbitrary Schwartz functions f, g and $T \in \text{GL}(\mathbb{R}^d)$, we have

$$\begin{aligned} (f * g)(Tx) &= \int_{\mathbb{R}^d} f(Tx - y) g(y) dy \\ &\stackrel{(y=Tz)}{=} |\det T| \cdot \int_{\mathbb{R}^d} f(Tx - Tz) g(Tz) dz \\ &= |\det T| \cdot [(f \circ T) * (g \circ T)](x), \end{aligned}$$

so that, if we understand the following equation componentwise,

$$\begin{aligned} (\nabla g_n) (T_i^T x) &= |\det T_i| \cdot [(g_n \circ T_i^T) * ([\nabla (\mathcal{F}^{-1}\psi)] \circ T_i^T)] (x) \\ (\text{with } \eta_j := \partial_j (\mathcal{F}^{-1}\psi)) &= |\det T_i| \cdot [(f_n * [\eta_j \circ T_i^T]) (x)]_{j \in \underline{d}}. \end{aligned} \quad (2.10)$$

Now, we divide the proof into the two cases $p \in [1, \infty]$ and $p \in (0, 1)$. In the (easier) case $p \in [1, \infty]$, we get

$$\begin{aligned} v(x) \cdot |(\partial_j g_n) (T_i^T x)| &= |\det T_i| \cdot v(x) \cdot |(f_n * [\eta_j \circ T_i^T]) (x)| \\ &\leq |\det T_i| \cdot \int_{\mathbb{R}^d} v(x) \cdot |f_n(y)| \cdot |\eta_j (T_i^T (x - y))| \, dy \\ (\text{eq. (2.7) and } \eta_j = \partial_j [\mathcal{F}^{-1}\psi]) &\leq C_1 \cdot |\det T_i| \cdot \int_{\mathbb{R}^d} v(x) \cdot |f_n(y)| \cdot (1 + |T_i^T (x - y)|)^{-N} \, dy \\ (\text{since } v(x) = v(x - y + y) \leq v(y) v_0(x - y)) &\leq C_1 \cdot |\det T_i| \cdot \int_{\mathbb{R}^d} |(v \cdot f_n)(y)| \cdot v_0(x - y) \cdot (1 + |T_i^T (x - y)|)^{-N} \, dy \\ (\text{assumption on } v_0 \text{ and eq. (1.11)}) &\leq \Omega_0^K \Omega_1 C_1 \cdot |\det T_i| \cdot \int_{\mathbb{R}^d} |(v \cdot f_n)(y)| \cdot (1 + |T_i^T (x - y)|)^{K-N} \, dy \end{aligned}$$

for arbitrary $x \in \mathbb{R}^d$.

But for $z \in T_i^{-T} [-2, 2]^d$, we have

$$\begin{aligned} 1 + |T_i^T x| &\leq 1 + |T_i^T (x + z)| + |-T_i^T z| \\ (|T_i^T z| \leq 2\sqrt{d} \text{ since } T_i^T z \in [-2, 2]^d) &\leq 1 + 2\sqrt{d} + |T_i^T (x + z)| \\ &\leq (1 + 2\sqrt{d}) (1 + |T_i^T (x + z)|) \end{aligned} \quad (2.11)$$

and

$$\begin{aligned} v(x) &= v(x + z - z) \leq v(x + z) \cdot v_0(-z) \\ &\leq \Omega_1 \cdot v(x + z) \cdot (1 + |-z|)^K \\ (\text{eq. (1.11)}) &\leq \Omega_0^K \Omega_1 \cdot v(x + z) \cdot (1 + |T_i^T z|)^K \\ &\leq \Omega_0^K \Omega_1 (1 + 2\sqrt{d})^K \cdot v(x + z). \end{aligned} \quad (2.12)$$

By applying these two estimates and noting $K - N < 0$, we get

$$\begin{aligned} v(x) \cdot |[(\partial_j g_n) \circ T_i^T] (x + z)| &\leq \Omega_0^K \Omega_1 \cdot (1 + 2\sqrt{d})^K \cdot v(x + z) \cdot |[(\partial_j g_n) \circ T_i^T] (x + z)| \\ &\leq \Omega_0^{2K} \Omega_1^2 C_1 \cdot (1 + 2\sqrt{d})^K \cdot |\det T_i| \cdot \int_{\mathbb{R}^d} |(v \cdot f_n)(y)| \cdot (1 + |T_i^T (x - y + z)|)^{K-N} \, dy \\ (\text{eq. (2.11) with } x - y \text{ instead of } x) &\leq \Omega_0^{2K} \Omega_1^2 C_1 \cdot (1 + 2\sqrt{d})^N \cdot |\det T_i| \cdot \int_{\mathbb{R}^d} |(v \cdot f_n)(y)| \cdot (1 + |T_i^T (x - y)|)^{K-N} \, dy. \end{aligned}$$

Noting that this holds for arbitrary $z \in T_i^{-T} [-2, 2]^d$ and by taking the ℓ^2 -norm over $j \in \underline{d}$, we conclude

$$\begin{aligned} v(x) \cdot [M_{T_i^{-T} [-2, 2]^d} ([\nabla g_n] \circ T_i^T)] (x) &\leq C_2 \cdot |\det T_i| \cdot \int_{\mathbb{R}^d} |(v \cdot f_n)(y)| \cdot (1 + |T_i^T (x - y)|)^{K-N} \, dy \\ &= C_2 \cdot |\det T_i| \cdot (|v \cdot f_n| * [(1 + |\bullet|)^{K-N} \circ T_i^T]) (x), \end{aligned}$$

where $C_2 := \Omega_0^{2K} \Omega_1^2 \cdot C_1 \sqrt{d} (1 + 2\sqrt{d})^N$.

By taking the L^p -norm and using Young's theorem for convolutions, we conclude

$$\begin{aligned} \|M_{T_i^{-T} [-2, 2]^d} ([\nabla g_n] \circ T_i^T)\|_{L_v^p} &\leq C_2 \cdot |\det T_i| \cdot \|v \cdot f_n\|_{L^p} \cdot \|(1 + |\bullet|)^{K-N} \circ T_i^T\|_{L^1} \\ &= C_2 \cdot \|f_n\|_{L_v^p} \cdot \|(1 + |\bullet|)^{K-N}\|_{L^1} \\ (\text{eq. (1.9) and } K - N = K - \lceil K + d + 1 \rceil \leq -(d + 1)) &\leq s_d C_2 \cdot \|f_n\|_{L_v^p} \\ (\text{since } |f_n| \leq |f|) &\leq s_d C_2 \cdot \|f\|_{L_v^p} < \infty. \end{aligned}$$

In view of equation (2.9), we have thus shown

$$\begin{aligned} \left\| \text{osc}_{\delta \cdot T_i^{-T}[-1,1]^d} f_n \right\|_{W_{T_i^{-T}[-1,1]^d}(L_v^p)} &\leq 2\sqrt{d} \cdot \delta \cdot \left\| M_{T_i^{-T}[-2,2]^d} [(\nabla g_n) \circ T_i^T] \right\|_{L_v^p} \\ &\leq 2s_d \sqrt{d} C_2 \cdot \delta \cdot \|f\|_{L_v^p} < \infty. \end{aligned}$$

Now, we note

$$\begin{aligned} 2s_d \sqrt{d} C_2 &= \Omega_0^{2K} \Omega_1^2 \cdot C_1 \cdot 2s_d \cdot d \cdot \left(1 + 2\sqrt{d}\right)^N \\ (\text{eq. (2.8)}) &\leq \Omega_0^{2K} \Omega_1^2 \cdot 2s_d \cdot d \cdot \left(1 + 2\sqrt{d}\right)^N \cdot 40 \cdot 10^d \cdot (96d \cdot N)^{N+1} \cdot (1+R)^{d+1} \\ (\text{since } s_d \leq 4^d \text{ and } d \leq 2^d) &\leq \Omega_0^{2K} \Omega_1^2 \cdot 2 \cdot 4^d 2^d \left(1 + 2\sqrt{d}\right)^N \cdot 40 \cdot 10^d \cdot (96d \cdot N)^{N+1} \cdot (1+R)^{d+1} \\ &\leq \Omega_0^{2K} \Omega_1^2 \cdot \left(1 + 2\sqrt{d}\right)^N \cdot 80^{d+1} \cdot (96d \cdot N)^{N+1} \cdot (1+R)^{d+1} \\ &\leq \Omega_0^{2K} \Omega_1^2 \cdot \left(3\sqrt{d}\right)^N \cdot 80^{d+1} \cdot (96d \cdot N)^{N+1} \cdot (1+R)^{d+1} \\ &\leq \Omega_0^{2K} \Omega_1^2 \cdot 80^{d+1} \cdot \left(288 \cdot d^{3/2} \cdot N\right)^{N+1} \cdot (1+R)^{d+1} \\ (\text{since } N \geq d+1) &\leq \Omega_0^{2K} \Omega_1^2 \cdot \left(23040 \cdot d^{3/2} \cdot N\right)^{N+1} \cdot (1+R)^{d+1} \\ (\text{def. of } N) &\leq \Omega_0^{2K} \Omega_1^2 \cdot \left(23040 \cdot d^{3/2} \cdot (K+d+2)\right)^{K+d+3} \cdot (1+R)^{d+1}. \end{aligned}$$

All that remains is to extend this estimate to f instead of f_n . But for $x \in \mathbb{R}^d$ and arbitrary $y, z \in \delta \cdot T_i^{-T}[-1,1]^d$, we have

$$|f(x+y) - f(x+z)| = \liminf_{n \rightarrow \infty} |f_n(x+y) - f_n(x+z)| \leq \liminf_{n \rightarrow \infty} \left(\text{osc}_{\delta \cdot T_i^{-T}[-1,1]^d} f_n \right)(x)$$

and thus $\left(\text{osc}_{\delta \cdot T_i^{-T}[-1,1]^d} f \right)(x) \leq \liminf_{n \rightarrow \infty} \left(\text{osc}_{\delta \cdot T_i^{-T}[-1,1]^d} f_n \right)(x)$ for all $x \in \mathbb{R}^d$. A similar estimate holds for the maximal function $M_{T_i^{-T}[-1,1]^d}$ instead of the oscillation. All in all, an application of Fatou's Lemma yields

$$\left\| \text{osc}_{\delta \cdot T_i^{-T}[-1,1]^d} f \right\|_{W_{T_i^{-T}[-1,1]^d}(L_v^p)} \leq \liminf_{n \rightarrow \infty} \left\| \text{osc}_{\delta \cdot T_i^{-T}[-1,1]^d} f_n \right\|_{W_{T_i^{-T}[-1,1]^d}(L_v^p)} \leq 2s_d \sqrt{d} C_2 \cdot \delta \cdot \|f\|_{L_v^p} < \infty,$$

as desired.

Now, we consider the case $p \in (0, 1)$. Here, we recall $\eta_j = \partial_j (\mathcal{F}^{-1} \psi)$ and observe

$$\begin{aligned} \text{supp } \mathcal{F}[\eta_j \circ T_i^T] &= \text{supp } \left[|\det T_i|^{-1} \cdot \widehat{\eta_j} \circ T_i^{-1} \right] \\ &= T_i \text{supp } \widehat{\eta_j} \\ &= T_i \text{supp } (\xi \mapsto 2\pi i \xi_j \cdot \mathcal{F}[\mathcal{F}^{-1} \psi](\xi)) \\ &\subset T_i \text{supp } \psi \subset T_i[-(R+4), R+4]^d \end{aligned}$$

and $\text{supp } \widehat{f_n} \subset T_i[-(R+1), R+1]^d$, so that equation (2.10) and Theorem 2.15 yield

$$\begin{aligned} &\left| (\partial_j g_n)(T_i^T x) \right|^p \\ &\leq |\det T_i|^p \cdot \left[\lambda_d \left(T_i \left([-(R+4), R+4]^d - [-(R+1), R+1]^d \right) \right) \right]^{1-p} \cdot \int_{\mathbb{R}^d} |f_n(y)|^p \cdot |(\eta_j \circ T_i^T)(x-y)|^p dy \\ (\text{eq. (2.7)}) &\leq |\det T_i| \cdot C_1^p [2 \cdot (2R+5)]^{d(1-p)} \cdot \int_{\mathbb{R}^d} |f_n(y)|^p \cdot (1 + |T_i^T(x-y)|)^{-Np} dy. \end{aligned}$$

Similar to the case $p \in [1, \infty]$, this implies

$$\begin{aligned} &|v(x) \cdot |(\partial_j g_n)(T_i^T x)||^p \leq |\det T_i| \cdot C_1^p [2 \cdot (2R+5)]^{d(1-p)} \cdot \int_{\mathbb{R}^d} |(v \cdot f_n)(y)|^p \cdot \left[v_0(x-y) \cdot (1 + |T_i^T(x-y)|)^{-N} \right]^p dy \\ (\text{assump. on } v_0 \text{ and eq. (1.11)}) &\leq [\Omega_0^K \Omega_1]^p \cdot |\det T_i| \cdot C_1^p [4R+10]^{d(1-p)} \cdot \int_{\mathbb{R}^d} |(v \cdot f_n)(y)|^p \cdot \left[(1 + |T_i^T(x-y)|)^{K-N} \right]^p dy. \end{aligned}$$

As above, equations (2.11) and (2.12) show for arbitrary $z \in T_i^{-T}[-2, 2]^d$ that

$$\begin{aligned}
 & [v(x) \cdot |[(\partial_j g_n) \circ T_i^T](x+z)|]^p \\
 (\text{eq. (2.12)}) & \leq \left[\Omega_0^K \Omega_1 \left(1 + 2\sqrt{d}\right)^K \right]^p \cdot |(v \cdot [(\partial_j g_n) \circ T_i^T])(x+z)|^p \\
 & \leq \left[\Omega_0^{2K} \Omega_1^2 \left(1 + 2\sqrt{d}\right)^K \right]^p \cdot |\det T_i| \cdot C_1^p (10+4R)^{d(1-p)} \cdot \int_{\mathbb{R}^d} |(v \cdot f_n)(y)|^p \cdot \left[(1 + |T_i^T(x+z-y)|)^{K-N} \right]^p dy \\
 (\text{eq. (2.11)}) & \leq \left[\Omega_0^{2K} \Omega_1^2 \left(1 + 2\sqrt{d}\right)^N \right]^p \cdot |\det T_i| \cdot C_1^p (10+4R)^{d(1-p)} \cdot \int_{\mathbb{R}^d} |(v \cdot f_n)(y)|^p \cdot (1 + |T_i^T(x-y)|)^{p(K-N)} dy.
 \end{aligned}$$

Since this holds for arbitrary $z \in T_i^{-T}[-2, 2]^d$ and $j \in \underline{d}$ and since

$$|[(\nabla g_n) \circ T_i^T](x+z)| \leq \sqrt{d} \cdot \max_{j \in \underline{d}} |[(\partial_j g_n) \circ T_i^T](x+z)|,$$

we conclude

$$\left[v \cdot M_{T_i^{-T}[-2, 2]^d}([(\nabla g_n) \circ T_i^T]) \right]^p \leq \left[\Omega_0^{2K} \Omega_1^2 \sqrt{d} \left(1 + 2\sqrt{d}\right)^N \right]^p \cdot |\det T_i| \cdot C_1^p (10+4R)^{d(1-p)} \cdot |v \cdot f_n|^p \cdot \left[(1 + |\bullet|)^{p(K-N)} \circ T_i^T \right],$$

so that Young's inequality $L^1 * L^1 \hookrightarrow L^1$ yields

$$\begin{aligned}
 \| [(\nabla g_n) \circ T_i^T] \|_{W_{T_i^{-T}[-2, 2]^d}(L_v^p)}^p & \leq \left[C_1 \Omega_0^{2K} \Omega_1^2 \sqrt{d} \left(1 + 2\sqrt{d}\right)^N \right]^p (10+4R)^{d(1-p)} \cdot |\det T_i| \cdot \|v \cdot f_n\|_{L^1}^p \cdot \left\| (1 + |\bullet|)^{p(K-N)} \circ T_i^T \right\|_{L^1} \\
 & = \left[\Omega_0^{2K} \Omega_1^2 \sqrt{d} \left(1 + 2\sqrt{d}\right)^N \right]^p \cdot C_1^p (10+4R)^{d(1-p)} \cdot \|f_n\|_{L_v^p}^p \cdot \left\| (1 + |\bullet|)^{p(K-N)} \right\|_{L^1} \\
 (\text{eq. (1.9) and } p(K-N) \leq -(d+1)) & \leq \left[\Omega_0^{2K} \Omega_1^2 \sqrt{d} \left(1 + 2\sqrt{d}\right)^N \right]^p \cdot C_1^p (10+4R)^{d(1-p)} \cdot \|f_n\|_{L_v^p}^p \cdot s_d \\
 (\text{since } |f_n| \leq |f| \text{ and } s_d \leq 4^d) & \leq \left[\Omega_0^{2K} \Omega_1^2 \sqrt{d} \left(1 + 2\sqrt{d}\right)^N 4^{d/p} C_1 (10+4R)^{d(\frac{1}{p}-1)} \right]^p \cdot \|f\|_{L_v^p}^p.
 \end{aligned}$$

In view of equation (2.9), we arrive at

$$\begin{aligned}
 \left\| \text{osc}_{\delta \cdot T_i^{-T}[-1, 1]^d} f_n \right\|_{W_{T_i^{-T}[-1, 1]^d}(L_v^p)} & \leq 2\sqrt{d} \cdot \delta \cdot \|(\nabla g_n) \circ T_i^T\|_{W_{T_i^{-T}[-2, 2]^d}(L_v^p)} \\
 & \leq \delta \cdot 2d \cdot \Omega_0^{2K} \Omega_1^2 \left(1 + 2\sqrt{d}\right)^N 4^{d/p} C_1 (10+4R)^{d(\frac{1}{p}-1)} \cdot \|f\|_{L_v^p}.
 \end{aligned}$$

The remainder of the proof is now as for $p \in [1, \infty]$, noting that

$$\begin{aligned}
 & 2d \cdot \Omega_0^{2K} \Omega_1^2 \left(1 + 2\sqrt{d}\right)^N 4^{d/p} \cdot C_1 (10+4R)^{d(\frac{1}{p}-1)} \\
 (\text{eq. (2.8)}) & \leq 2d \cdot \Omega_0^{2K} \Omega_1^2 \left(1 + 2\sqrt{d}\right)^N 4^{d/p} \cdot 40 \cdot 10^d \cdot (96d \cdot N)^{N+1} \cdot 10^{d(\frac{1}{p}-1)} (1+R)^{1+\frac{d}{p}} \\
 (\text{since } d \leq 2^d \leq 2^{d/p}) & \leq \Omega_0^{2K} \Omega_1^2 \cdot \left(3\sqrt{d}\right)^N \cdot (96d \cdot N)^{N+1} \cdot 80^{1+\frac{d}{p}} (1+R)^{1+\frac{d}{p}} \\
 (\text{since } N+1 \geq \frac{d+1}{p} + 1 \geq \frac{d}{p} + 1) & \leq \Omega_0^{2K} \Omega_1^2 \cdot \left(23040 \cdot d^{3/2} \cdot N\right)^{N+1} \cdot (1+R)^{1+\frac{d}{p}} \\
 (\text{since } N \leq K + \frac{d+1}{p} + 1) & \leq \Omega_0^{2K} \Omega_1^2 \cdot \left(23040 \cdot d^{3/2} \cdot \left(K + 1 + \frac{d+1}{p}\right)\right)^{K+2+\frac{d+1}{p}} \cdot (1+R)^{1+\frac{d}{p}}. \quad \square
 \end{aligned}$$

2.4. Convolution relation for Wiener amalgam spaces. Finally, we come to the convolution relation for the Wiener Amalgam spaces. The theorem stated here is a slight variation (and specialization) of [76, Theorem 2.3.24], which originally appeared in [68].

Theorem 2.19. Let $Q_1, Q_2 \subset \mathbb{R}^d$ be bounded, Borel measurable unit neighborhoods and assume that $Q_1 - Q_1$ and $Q_2 - Q_1$ are measurable. Let $p \in (0, \infty]$ and set $r := \min\{1, p\}$.

Assume that there is a countable family $(x_i)_{i \in I}$ in \mathbb{R}^d satisfying $\mathbb{R}^d = \bigcup_{i \in I} (x_i + Q_1)$ and

$$N := \sup_{x \in \mathbb{R}^d} |\{i \in I \mid x \in x_i + Q_1\}| < \infty.$$

Then we have for every $f \in W_{Q_1-Q_1}(L_{v_0}^r)$ and every $g \in W_{Q_2-Q_1}(L_v^p)$ that

- $f \in L_{v_0}^1(\mathbb{R}^d)$ with $\|f\|_{L_{v_0}^1} \lesssim \|f\|_{W_{Q_1-Q_1}(L_{v_0}^r)}$, where the implied constant only depends on N, Q_1, r, v_0 .
 - $g \in L_v^\infty(\mathbb{R}^d)$ with $\|g\|_{L_v^\infty} \lesssim \|g\|_{W_{Q_2-Q_1}(L_v^p)}$, where the implied constant only depends on N, Q_1, Q_2, p, v_0 .
- In particular,

$$W_{Q_2-Q_1}(L_v^p) \hookrightarrow L_v^\infty(\mathbb{R}^d) \hookrightarrow L_{(1+|\bullet|)^{-\kappa}}^\infty(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d).$$

- The convolution $f * g : \mathbb{R}^d \rightarrow \mathbb{C}$ is a well-defined continuous function.
- We have

$$\|f * g\|_{W_{Q_2}(L_v^p)} \leq N^{\frac{1}{r}} \cdot \left[\sup_{x \in Q_1} v_0(x) \right] \cdot [\lambda_d(Q_1)]^{1-\frac{1}{r}} \cdot \|f\|_{W_{Q_1-Q_1}(L_{v_0}^r)} \cdot \|g\|_{W_{Q_2-Q_1}(L_v^p)}. \quad \blacktriangleleft$$

Remark 2.20. Since one can always choose a compact unit neighborhood Q_1 for which the assumptions of the theorem are satisfied (choose e.g. $Q_1 = [-\frac{1}{2}, \frac{1}{2}]^d$ and $x_i := i$ for $i \in I := \mathbb{Z}^d$), we see in view of Lemma 2.7 that

$$W_Q(L_v^p) \hookrightarrow L_v^\infty(\mathbb{R}^d) \hookrightarrow L_{(1+|\bullet|)^{-\kappa}}^\infty(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d) \quad (2.13)$$

holds for every bounded unit-neighborhood $Q \subset \mathbb{R}^d$. The same also holds with v instead of v_0 , since v_0 satisfies all properties that v has. \blacklozenge

Proof. In the following, we will frequently use the discrete weights $v_i^{\text{disc}} := v(x_i)$ and $(v_0^{\text{disc}})_i := v_0(x_i)$, as well as the constant $C_1 := \sup_{x \in Q_1} v_0(x) \leq \Omega_1 \cdot \sup_{x \in Q_1} (1 + |x|)^K < \infty$, which is finite since Q_1 is bounded. These quantities are important, since we have for $y = x_i + q \in x_i + Q_1$ the estimates

$$v(y) = v(x_i + q) \leq v(x_i) v_0(q) \leq C_1 \cdot v_i^{\text{disc}} \quad \text{and similarly} \quad v_0(y) \leq C_1 \cdot (v_0^{\text{disc}})_i.$$

Likewise, by symmetry and submultiplicativity of v_0 , we also have

$$(v_0^{\text{disc}})_i = v_0(x_i) = v_0(y - q) \leq v_0(y) \cdot v_0(-q) = v_0(y) \cdot v_0(q) \leq C_1 \cdot v_0(y).$$

Completely similar, we also get $v_i^{\text{disc}} \leq C_1 \cdot v(y)$ for all $y \in x_i + Q_1$.

Now, we first show that we can write each $h \in W_{Q_1-Q_1}(L_{v_0}^r)$ as $h = \sum_{i \in I} h_i$, where³ $\text{supp } h_i \subset x_i + Q_1$ and where

$$\|(\|h_i\|_{L^\infty})_{i \in I}\|_{\ell_{v_0^{\text{disc}}}^r} \leq C_1 \cdot \frac{N^{1/r}}{[\lambda_d(Q_1)]^{1/r}} \cdot \|h\|_{W_{Q_1-Q_1}(L_{v_0}^r)}. \quad (2.14)$$

Indeed, since I is countable (and necessarily infinite, since $\mathbb{R}^d = \bigcup_{i \in I} (x_i + Q_1)$, with Q_1 bounded), we can assume $I = \mathbb{N}$. Then, define $h_i := h \cdot \mathbf{1}_{(x_i + Q_1) \setminus \bigcup_{j=1}^{i-1} (x_j + Q_1)}$. Because of $\mathbb{R}^d = \bigcup_{i \in I} (x_i + Q_1)$, this easily yields $h = \sum_{i \in I} h_i$ and $\text{supp } h_i \subset x_i + Q_1$ is trivial, so that we only need to verify estimate (2.14).

To this end, first note for $x \in x_i + Q_1$ that $x_i \in x - Q_1$ and hence $x_i + Q_1 \subset x + Q_1 - Q_1$, which yields

$$\|h_i\|_{L^\infty} \leq \|h \cdot \mathbf{1}_{x_i + Q_1}\|_{L^\infty} \leq \|h \cdot \mathbf{1}_{x + Q_1 - Q_1}\|_{L^\infty} = (M_{Q_1-Q_1} h)(x).$$

Now, take the r -th power of this estimate, multiply both sides with $v_0^r(x) \cdot \mathbf{1}_{x_i + Q_1}(x)$ and sum over $i \in I$ to arrive at

$$\sum_{i \in I} [\|h_i\|_{L^\infty}^r \cdot v_0^r(x) \cdot \mathbf{1}_{x_i + Q_1}(x)] \leq [v_0(x) \cdot (M_{Q_1-Q_1} h)(x)]^r \cdot \sum_{i \in I} \mathbf{1}_{x_i + Q_1}(x) \leq N \cdot [v_0(x) \cdot (M_{Q_1-Q_1} h)(x)]^r.$$

As observed at the beginning of the proof, we have $\mathbf{1}_{x_i + Q_1} \cdot (v_0^{\text{disc}})_i \leq C_1 \cdot v_0 \cdot \mathbf{1}_{x_i + Q_1}$. By combining this with the preceding estimate and integrating, we get

$$\begin{aligned} \lambda_d(Q_1) \cdot \|(\|h_i\|_{L^\infty})_{i \in I}\|_{\ell_{v_0^{\text{disc}}}^r}^r &= \sum_{i \in I} \|h_i\|_{L^\infty}^r \cdot (v_0^{\text{disc}})_i^r \cdot \lambda_d(x_i + Q_1) \\ &= \int_{\mathbb{R}^d} \sum_{i \in I} (\|h_i\|_{L^\infty} \cdot (v_0^{\text{disc}})_i \cdot \mathbf{1}_{x_i + Q_1}(x))^r dx \\ &\leq C_1^r N \cdot \int_{\mathbb{R}^d} [v_0(x) \cdot (M_{Q_1-Q_1} h)(x)]^r dx \\ &= C_1^r N \cdot \|h\|_{W_{Q_1-Q_1}(L_{v_0}^r)}^r < \infty. \end{aligned}$$

Rearranging shows that equation (2.14) is indeed satisfied.

³In this proof and the next, but not elsewhere in the paper, we write $\text{supp } f := \{x \in \mathbb{R}^d \mid f(x) \neq 0\}$, which is different from the usual meaning $\text{supp } f := \overline{\{x \in \mathbb{R}^d \mid f(x) \neq 0\}}$.

Now, since $\ell^r(I) \hookrightarrow \ell^1(I)$ and because of $\text{supp } h_i \subset x_i + Q_1$, so that

$$v_0 \cdot |h_i| \leq C_1 \cdot (v_0^{\text{disc}})_i \cdot |h_i| \leq C_1 \cdot (v_0^{\text{disc}})_i \cdot \|h_i\|_{L^\infty} \cdot \mathbf{1}_{x_i+Q_1} \quad \text{almost everywhere,}$$

we get

$$\begin{aligned} \|h\|_{L^1_{v_0}} &\leq \sum_{i \in I} \|h_i\|_{L^1_{v_0}} \leq \left[\sum_{i \in I} \|h_i\|_{L^1_{v_0}}^r \right]^{1/r} \\ &\leq C_1 \cdot \left[\sum_{i \in I} (v_0^{\text{disc}})_i^r \cdot \|h_i\|_{L^\infty}^r \cdot [\lambda(x_i + Q_1)]^r \right]^{1/r} \\ &\leq C_1 \cdot \lambda_d(Q_1) \cdot \|(\|h_i\|_{L^\infty})_{i \in I}\|_{\ell^r_{v_0^{\text{disc}}}} \\ &\quad (\text{eq. (2.14)}) \leq C_1^2 \cdot [\lambda_d(Q_1)]^{1-\frac{1}{r}} \cdot N^{1/r} \cdot \|h\|_{W_{Q_1-Q_1}(L^r_{v_0})} < \infty, \end{aligned}$$

which proves the first part of the theorem.

Now, we want to prove the second part of the theorem. For $p = \infty$, we have $\|g\|_{L^\infty_v} = \|g\|_{L^p_v} \leq \|g\|_{W_{Q_2-Q_1}(L^p_v)}$ by Lemma 2.2, so that we can assume $p \in (0, \infty)$.

Next, we define $g_i := g \cdot \mathbf{1}_{x_i+Q_1}$ for $i \in I$ and note for $x \in x_i + Q_1$ as above that $x_i + Q_1 \subset x + Q_1 - Q_1$, so that

$$\|g_i\|_{L^\infty} \leq \|g \cdot \mathbf{1}_{x+Q_1-Q_1}\|_{L^\infty} = (M_{Q_1-Q_1}g)(x).$$

Hence,

$$\begin{aligned} \frac{1}{C_1} \cdot v_i^{\text{disc}} \cdot \|g_i\|_{L^\infty} \cdot [\lambda_d(Q_1)]^{1/p} &= \frac{1}{C_1} \cdot \left[\int_{x_i+Q_1} (v_i^{\text{disc}} \cdot \|g_i\|_{L^\infty})^p dx \right]^{1/p} \\ &\leq \left[\int_{\mathbb{R}^d} (v(x) \cdot \|g_i\|_{L^\infty} \cdot \mathbf{1}_{x_i+Q_1}(x))^p dx \right]^{1/p} \\ &\leq \left(\int_{\mathbb{R}^d} [(v \cdot M_{Q_1-Q_1}g)(x)]^p dx \right)^{1/p} \\ &= \|g\|_{W_{Q_1-Q_1}(L^p_v)} \\ &\quad (\text{Lemma 2.7}) \leq C_2 \cdot \|g\|_{W_{Q_2-Q_1}(L^p_v)}. \end{aligned}$$

Here, the last step used that $Q_1 - Q_1$ and $Q_2 - Q_1$ are both measurable, bounded unit-neighborhoods, so that Lemma 2.7 yields a constant $C_2 = C_2(Q_1, Q_2, v_0, p) > 0$ satisfying $\|g\|_{W_{Q_1-Q_1}(L^p_v)} \leq C_2 \cdot \|g\|_{W_{Q_2-Q_1}(L^p_v)}$.

But there is a null-set $N_i \subset x_i + Q_1$ satisfying $|g(x)| = |g_i(x)| \leq \|g_i\|_{L^\infty}$ for all $x \in (x_i + Q_1) \setminus N_i$. Hence,

$$v(x) \cdot |g(x)| \leq C_1 \cdot v_i^{\text{disc}} \cdot \|g_i\|_{L^\infty} \leq \frac{C_1^2 C_2}{[\lambda_d(Q_1)]^{1/p}} \cdot \|g\|_{W_{Q_2-Q_1}(L^p_v)}$$

for all $x \in (x_i + Q_1) \setminus N_i$. But since $N := \bigcup_{i \in I} N_i \subset \mathbb{R}^d$ is a null-set and since $\mathbb{R}^d = \bigcup_{i \in I} (x_i + Q_1)$, we get $\|g\|_{L^\infty_v} \leq \frac{C_1^2 C_2}{[\lambda_d(Q_1)]^{1/p}} \cdot \|g\|_{W_{Q_2-Q_1}(L^p_v)}$, which proves the main part of the second part of the theorem for $p \in (0, \infty)$.

To establish the embedding $W_{Q_2-Q_1}(L^p_v) \hookrightarrow L^\infty_v(\mathbb{R}^d) \hookrightarrow L^\infty_{(1+|\bullet|)^{-\kappa}}(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$, we first observe that $L^\infty_{(1+|\bullet|)^{-\kappa}}(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$ is trivial. Furthermore,

$$v(0) = v(x + (-x)) \leq v(x) \cdot v_0(-x) \leq \Omega_1 (1 + |-x|)^K \cdot v(x) \quad \forall x \in \mathbb{R}^d, \quad (2.15)$$

so that $v(x) \geq \frac{v(0)}{\Omega_1} \cdot (1 + |x|)^{-K}$ and hence $W_{Q_2-Q_1}(L^p_v) \hookrightarrow L^\infty_v(\mathbb{R}^d) \hookrightarrow L^\infty_{(1+|\bullet|)^{-\kappa}}(\mathbb{R}^d)$, as desired.

Now, note for $f \in L^1_{v_0}(\mathbb{R}^d)$ and $g \in L^\infty_v(\mathbb{R}^d)$ because of $v(x) = v(y + (x - y)) \leq v(y) \cdot v_0(x - y)$ that

$$\begin{aligned} v(x) \cdot \int_{\mathbb{R}^d} |f(x - y)| \cdot |g(y)| dy &\leq \int_{\mathbb{R}^d} |(v_0 \cdot f)(x - y)| \cdot |(v \cdot g)(y)| dy \\ &\leq \|g\|_{L^\infty_v} \cdot \|f\|_{L^1_{v_0}} < \infty \quad \forall x \in \mathbb{R}^d. \end{aligned} \quad (2.16)$$

Hence, $(f * g)(x)$ is well-defined for all $x \in \mathbb{R}^d$ and $\|f * g\|_{L^\infty_{(1+|\bullet|)^{-\kappa}}} \lesssim \|f * g\|_{L^\infty_v} \leq \|f\|_{L^1_{v_0}} \cdot \|g\|_{L^\infty_v}$. Now, note that the subspace $C(\mathbb{R}^d) \cap L^\infty_{(1+|\bullet|)^{-\kappa}}(\mathbb{R}^d)$ of continuous functions in $L^\infty_{(1+|\bullet|)^{-\kappa}}(\mathbb{R}^d)$ is a closed subspace of

$L_{(1+|\bullet|)^{-\kappa}}^\infty(\mathbb{R}^d)$. Furthermore, $C_c(\mathbb{R}^d) \subset L_{v_0}^1(\mathbb{R}^d)$ is dense and $L_v^\infty(\mathbb{R}^d) \hookrightarrow L_{(1+|\bullet|)^{-\kappa}}^\infty(\mathbb{R}^d) \hookrightarrow L_{\text{loc}}^\infty(\mathbb{R}^d)$. But for $f \in C_c(\mathbb{R}^d)$ and $g \in L_{\text{loc}}^\infty(\mathbb{R}^d)$, it is not hard to see that $f * g$ is continuous.

Altogether, the preceding properties show that $f * g \in C(\mathbb{R}^d) \cap L_{(1+|\bullet|)^{-\kappa}}^\infty(\mathbb{R}^d)$ is well-defined and continuous for all $f \in L_{v_0}^1(\mathbb{R}^d)$ and $g \in L_v^\infty(\mathbb{R}^d)$. But in the setting of the theorem, we have $f \in W_{Q_1-Q_1}(L_{v_0}^r) \hookrightarrow L_{v_0}^1(\mathbb{R}^d)$ and $g \in W_{Q_2-Q_1}(L_v^p) \hookrightarrow L_v^\infty(\mathbb{R}^d)$, so that the third part of the theorem is established.

It remains to prove the last part of the theorem. To this end, recall from equation (2.14) that we can write $f = \sum_{i \in I} f_i$, where $\text{supp } f_i \subset x_i + Q_1$ and such that equation (2.14) is fulfilled, with f_i instead of h_i and f instead of h .

Next, we estimate $M_{Q_2}(f_i * g)$ for each $i \in I$ as follows: For $x \in \mathbb{R}^d$ and $q \in Q_2$, we have

$$\begin{aligned} |(f_i * g)(x + q)| &\leq (|f_i| * |g|)(x + q) = \int_{\mathbb{R}^d} |f_i(y)| \cdot |g(x + q - y)| \, dy \\ &\leq \|f_i\|_{L^\infty} \cdot \int_{\mathbb{R}^d} \mathbb{1}_{x_i+Q_1}(y) \cdot |g(x + q - y)| \, dy \\ &\stackrel{(z=x+q-y)}{=} \|f_i\|_{L^\infty} \cdot \int_{\mathbb{R}^d} \mathbb{1}_{x_i+Q_1}(x + q - z) \cdot |g(z)| \, dz \\ &\stackrel{(x+q-z \in x_i+Q_1 \text{ implies } z \in x-x_i+q-Q_1 \subset x-x_i+Q_2-Q_1)}{\leq} \|f_i\|_{L^\infty} \cdot \int_{\mathbb{R}^d} \mathbb{1}_{x_i+Q_1}(x + q - z) \, dz \cdot \|g \cdot \mathbb{1}_{x-x_i+Q_2-Q_1}\|_{L^\infty} \\ &= \|f_i\|_{L^\infty} \cdot \lambda_d(x + q - x_i - Q_1) \cdot (M_{Q_2-Q_1}g)(x - x_i) \\ &= \lambda_d(Q_1) \cdot \|f_i\|_{L^\infty} \cdot (L_{x_i}[M_{Q_2-Q_1}g])(x). \end{aligned}$$

Since this holds for all $q \in Q_2$, we get

$$[M_{Q_2}(|f_i| * |g|)](x) \leq \lambda_d(Q_1) \cdot \|f_i\|_{L^\infty} \cdot (L_{x_i}[M_{Q_2-Q_1}g])(x) \quad \forall x \in \mathbb{R}^d.$$

In view of Lemma 2.6 and by solidity of $L_v^p(\mathbb{R}^d)$, this implies

$$\begin{aligned} \|M_{Q_2}[|f_i| * |g|]\|_{L_v^p} &\leq \|f_i\|_{L^\infty} \cdot \lambda_d(Q_1) \cdot \|L_{x_i}[M_{Q_2-Q_1}g]\|_{L_v^p} \\ &\leq (v_0^{\text{disc}})_i \cdot \|f_i\|_{L^\infty} \cdot \lambda_d(Q_1) \cdot \|g\|_{W_{Q_2-Q_1}(L_v^p)}. \end{aligned}$$

Next, it is not hard to see $M_{Q_2}(\sum_{i \in I} h_i) \leq \sum_{i \in I} M_{Q_2}h_i$, so that we get because of

$$|(f * g)(x)| \leq (|f| * |g|)(x) \leq \sum_{i \in I} (|f_i| * |g|)(x)$$

that

$$\begin{aligned} \|M_{Q_2}(f * g)\|_{L_v^p}^r &\leq \|M_{Q_2}[|f| * |g|]\|_{L_v^p}^r \leq \left\| \sum_{i \in I} M_{Q_2}[|f_i| * |g|] \right\|_{L_v^p}^r \\ &\stackrel{(L_v^p \text{ satisfies the } r\text{-triangle inequality})}{\leq} \sum_{i \in I} \|M_{Q_2}[|f_i| * |g|]\|_{L_v^p}^r \\ &\leq \left[\lambda_d(Q_1) \cdot \|g\|_{W_{Q_2-Q_1}(L_v^p)} \right]^r \cdot \sum_{i \in I} (v_0^{\text{disc}})_i^r \cdot \|f_i\|_{L^\infty}^r \\ &\stackrel{(\text{eq. (2.14)})}{\leq} \left[\lambda_d(Q_1) \cdot \|g\|_{W_{Q_2-Q_1}(L_v^p)} \right]^r \cdot \left(C_1 \cdot \frac{N^{1/r}}{[\lambda_d(Q_1)]^{1/r}} \cdot \|f\|_{W_{Q_1-Q_1}(L_{v_0}^r)} \right)^r, \end{aligned}$$

which finally yields

$$\|f * g\|_{W_{Q_2}(L_v^p)} \leq N^{\frac{1}{r}} C_1 \cdot [\lambda_d(Q_1)]^{1-\frac{1}{r}} \cdot \|f\|_{W_{Q_1-Q_1}(L_{v_0}^r)} \cdot \|g\|_{W_{Q_2-Q_1}(L_v^p)} < \infty,$$

as desired. \square

With a very slight variant of the above proof, one can also show the following modification of the theorem. For completeness, we provide the proof, but with slightly less details than above.

Proposition 2.21. *Under the assumptions of Theorem 2.19, if $p \in (0, 1]$, then*

$$\|f * g\|_{W_{Q_2}(L_v^p)} \leq N^{\frac{1}{p}} \cdot \left[\sup_{x \in Q_1} v_0(x) \right] \cdot [\lambda_d(Q_1)]^{1-\frac{1}{p}} \cdot \|f\|_{W_{Q_2-Q_1}(L_{v_0}^p)} \cdot \|g\|_{W_{Q_1-Q_1}(L_v^p)}. \quad \blacktriangleleft$$

Proof. As in the proof of Theorem 2.19, let $C_1 := \sup_{x \in Q_1} v_0(x)$. Also as in that proof, we can assume $I = \mathbb{N}$, so that we have $g = \sum_{i \in I} g_i$ with $\text{supp } g_i \subset x_i + Q_1$ for $g_i := g \cdot \mathbb{1}_{(x_i + Q_1) \setminus \bigcup_{j=1}^{i-1} (x_j + Q_1)}$. Furthermore, for arbitrary $x \in x_i + Q_1$, we have $x_i + Q_1 \subset x + Q_1 - Q_1$ and thus

$$\|g_i\|_{L^\infty} \leq \|g \cdot \mathbb{1}_{x_i + Q_1}\|_{L^\infty} \leq \|g \cdot \mathbb{1}_{x + Q_1 - Q_1}\|_{L^\infty} = (M_{Q_1 - Q_1} g)(x).$$

Now, multiply both sides with $v(x)$, take the p th power, multiply with $\mathbb{1}_{x_i + Q_1}(x)$ and sum over $i \in I$ to obtain

$$\begin{aligned} \sum_{i \in I} [v(x) \cdot \|g_i\|_{L^\infty}]^p \mathbb{1}_{x_i + Q_1}(x) &\leq \sum_{i \in I} [v(x) \cdot (M_{Q_1 - Q_1} g)(x)]^p \mathbb{1}_{x_i + Q_1}(x) \\ &\leq N \cdot [v(x) \cdot (M_{Q_1 - Q_1} g)(x)]^p. \end{aligned}$$

But for $x \in x_i + Q_1$, i.e., $x = x_i + q$ with $q \in Q_1$, we have

$$v(x_i) = v(x - q) \leq v(x) \cdot v_0(-q) = v(x) \cdot v_0(q) \leq C_1 \cdot v(x),$$

so that we arrive at

$$\sum_{i \in I} [v(x_i) \cdot \|g_i\|_{L^\infty}]^p \mathbb{1}_{x_i + Q_1}(x) \leq C_1^p \cdot N \cdot [v(x) \cdot (M_{Q_1 - Q_1} g)(x)]^p.$$

Integrating this estimate over $x \in \mathbb{R}^d$ finally yields

$$\lambda_d(Q_1) \cdot \sum_{i \in I} [v(x_i) \cdot \|g_i\|_{L^\infty}]^p \leq C_1^p \cdot N \cdot \|g\|_{W_{Q_1 - Q_1}(L_{v_0}^p)}^p < \infty. \quad (2.17)$$

Now, let $x \in \mathbb{R}^d$ and $q \in Q_2$ be arbitrary. Since $\text{supp } g_i \subset x_i + Q_1$, we have

$$\begin{aligned} (|f| * |g_i|)(x + q) &\leq \|g_i\|_{L^\infty} \cdot \int_{\mathbb{R}^d} \mathbb{1}_{x_i + Q_1}(y) \cdot |f(x + q - y)| \, dy \\ (z = x + q - y) &= \|g_i\|_{L^\infty} \cdot \int_{\mathbb{R}^d} \mathbb{1}_{x_i + Q_1}(x + q - z) \cdot |f(z)| \, dz \\ (x + q - z \in x_i + Q_1 \text{ implies } z \in x + q - x_i - Q_1 \subset x - x_i + Q_2 - Q_1) &\leq \|g_i\|_{L^\infty} \cdot \int_{\mathbb{R}^d} \mathbb{1}_{x_i + Q_1}(x + q - z) \, dz \cdot \|f \cdot \mathbb{1}_{x - x_i + Q_2 - Q_1}\|_{L^\infty} \\ &\leq \|g_i\|_{L^\infty} \cdot \lambda_d(x + q - x_i - Q_1) \cdot \|f \cdot \mathbb{1}_{x - x_i + Q_2 - Q_1}\|_{L^\infty} \\ &= \lambda_d(Q_1) \cdot \|g_i\|_{L^\infty} \cdot (M_{Q_2 - Q_1} f)(x - x_i). \end{aligned}$$

Since this holds for arbitrary $q \in Q_2$, we have shown

$$[M_{Q_2}(|f| * |g_i|)](x) \leq \lambda_d(Q_1) \cdot \|g_i\|_{L^\infty} \cdot (M_{Q_2 - Q_1} f)(x - x_i).$$

Hence,

$$\begin{aligned} v(x) \cdot [M_{Q_2}(|f| * |g_i|)](x) &\leq \lambda_d(Q_1) \cdot \|g_i\|_{L^\infty} \cdot v(x) \cdot (M_{Q_2 - Q_1} f)(x - x_i) \\ (\text{since } v(x) = v(x - x_i + x_i) \leq v_0(x - x_i) v(x_i)) &\leq \lambda_d(Q_1) \cdot v(x_i) \|g_i\|_{L^\infty} \cdot [v_0 \cdot M_{Q_2 - Q_1} f](x - x_i). \end{aligned}$$

Taking the L^p norm on both sides, and using the isometric translation invariance of L^p , we conclude

$$\| |f| * |g_i| \|_{W_{Q_2}(L_{v_0}^p)} \leq \lambda_d(Q_1) \cdot v(x_i) \|g_i\|_{L^\infty} \cdot \|f\|_{W_{Q_2 - Q_1}(L_{v_0}^p)}.$$

Now, we finally combine the estimate $|(f * g)(x)| \leq (|f| * |g|)(x) \leq \sum_{i \in I} (|f| * |g_i|)(x)$ with solidity of $W_{Q_2}(L_{v_0}^p)$ and with the p -triangle inequality for $W_{Q_2}(L_{v_0}^p)$ (which holds since $p \in (0, 1]$) to deduce

$$\begin{aligned} \|f * g\|_{W_{Q_2}(L_{v_0}^p)}^p &\leq [\lambda_d(Q_1)]^p \cdot \|f\|_{W_{Q_2 - Q_1}(L_{v_0}^p)}^p \cdot \sum_{i \in I} [v(x_i) \|g_i\|_{L^\infty}]^p \\ (\text{eq. (2.17)}) &\leq [\lambda_d(Q_1)]^{p-1} \cdot C_1^p \cdot N \cdot \|f\|_{W_{Q_2 - Q_1}(L_{v_0}^p)}^p \cdot \|g\|_{W_{Q_1 - Q_1}(L_{v_0}^p)}^p, \end{aligned}$$

which easily yields the claim. \square

We now formulate an important special case of Theorem 2.19 as a corollary:

Corollary 2.22. *Let $i, j \in I$, $p \in (0, \infty]$, $f \in W_{T_j^{-T}[-1, 1]^d}(L_{v_0}^r)$ for $r := \min\{1, p\}$ and $g \in W_{T_j^{-T}[-1, 1]^d}(L_v^p)$. Then the convolution $f * g$ is pointwise defined and continuous and we have*

$$\|f * g\|_{W_{T_j^{-T}[-1, 1]^d}(L_v^p)} \leq \Omega_0^{3K} \Omega_1^3 C \cdot |\det T_j|^{\frac{1}{r}-1} \cdot \|f\|_{W_{T_j^{-T}[-1, 1]^d}(L_{v_0}^r)} \cdot \|g\|_{W_{T_j^{-T}[-1, 1]^d}(L_v^p)}$$

for $C := d^{-\frac{d}{2r}} \cdot (972 \cdot d^{5/2})^{K + \frac{d}{r}}$. \blacktriangleleft

Proof. We apply Theorem 2.19 with $Q_1 = Q_2 = T_j^{-T}[-1, 1]^d$. Note that we have

$$\mathbb{R} = \bigcup_{k \in \mathbb{Z}} (2k + [-1, 1]) \quad \text{and hence} \quad \mathbb{R}^d = \bigcup_{k \in \mathbb{Z}^d} (2k + [-1, 1]^d).$$

Furthermore, if $x \in (2k + [-1, 1]^d) \cap (2\ell + [-1, 1]^d)$, we get $2k + \mu = x = 2\ell + \nu$ for certain $\mu, \nu \in [-1, 1]^d$ and thus $\|k - \ell\|_\infty = \|\frac{\nu - \mu}{2}\|_\infty \leq 1$. Thus, we see (by fixing $k \in \mathbb{Z}^d$ with $x \in 2k + [-1, 1]^d$) that $x \in 2\ell + [-1, 1]^d$ can hold for at most 3^d values of ℓ , namely for $\ell \in \prod_{j=1}^d \{k_j - 1, k_j, k_j + 1\}$. Since $T_j^{-T} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is bijective, we see

$$\mathbb{R}^d = \bigcup_{k \in \mathbb{Z}^d} (2T_j^{-T}k + T_j^{-T}[-1, 1]^d) \quad \text{and} \quad N := \sup_{x \in \mathbb{R}^d} \left| \left\{ \ell \in \mathbb{Z}^d \mid x \in 2T_j^{-T}\ell + T_j^{-T}[-1, 1]^d \right\} \right| \leq 3^d. \quad (2.18)$$

Furthermore, equation (1.11) yields

$$\begin{aligned} \sup_{x \in Q_1} v_0(x) &= \sup_{y \in [-1, 1]^d} v_0(T_j^{-T}y) \leq \Omega_1 \cdot \sup_{y \in [-1, 1]^d} (1 + |T_j^{-T}y|)^K \\ &\stackrel{(\text{eq. (1.11)})}{\leq} \Omega_0^K \Omega_1 \cdot \sup_{y \in [-1, 1]^d} (1 + |y|)^K \\ &\leq (2\sqrt{d})^K \Omega_0^K \Omega_1. \end{aligned}$$

All in all, Theorem 2.19 shows

$$\begin{aligned} \|f * g\|_{W_{T_j^{-T}[-1, 1]^d}(L_v^p)} &= \|f * g\|_{W_{Q_2}(L_v^p)} \\ &\leq 3^{\frac{d}{2}} \cdot (2\sqrt{d})^K \Omega_0^K \Omega_1 \cdot [\lambda_d(Q_1)]^{1-\frac{1}{r}} \cdot \|f\|_{W_{Q_1-Q_1}(L_{v_0}^r)} \cdot \|g\|_{W_{Q_2-Q_1}(L_v^p)} \\ (Q_2-Q_1=Q_1-Q_1=T_j^{-T}[-2, 2]^d) &\leq 2^K 3^{\frac{d}{2}} \cdot d^{\frac{K}{2}} \cdot 2^{d(1-\frac{1}{r})} \cdot \Omega_0^K \Omega_1 \cdot |\det T_j^{-T}|^{1-\frac{1}{r}} \cdot \|f\|_{W_{T_j^{-T}[-2, 2]^d}(L_{v_0}^r)} \cdot \|g\|_{W_{T_j^{-T}[-2, 2]^d}(L_v^p)} \\ &\stackrel{(\text{eq. (2.2)})}{\leq} \Omega_0^{3K} \Omega_1^3 \cdot d^{-\frac{d}{2r}} \cdot (972 \cdot d^{\frac{5}{2}})^{K+\frac{d}{r}} \cdot |\det T_j|^{\frac{1}{r}-1} \cdot \|f\|_{W_{T_j^{-T}[-1, 1]^d}(L_{v_0}^r)} \cdot \|g\|_{W_{T_j^{-T}[-1, 1]^d}(L_v^p)} \cdot \square \end{aligned}$$

Next, we establish a more quantitative—and weighted—version of the convolution relation for (suitably) bandlimited functions given in [77, Corollary 3.14], which is in turn a specialized version of [73, Proposition in §1.5.1].

The following proposition uses the notation $Q_i^{n*} := \bigcup_{\ell \in i^{n*}} Q_\ell$, where $i^{1*} := i^*$ and $i^{(n+1)*} := \bigcup_{\ell \in i^{n*}} \ell^*$. For the definition of i^* , cf. equation (1.10).

Proposition 2.23. *Let $p \in (0, 1]$ and $n \in \mathbb{N}_0$. If $i \in I$ and*

- *if $\psi \in C_c^\infty(\mathbb{R}^d)$ with $\text{supp } \psi \subset \overline{Q_i^{n*}}$ and*
- *if $f \in \mathcal{D}'(\mathcal{O})$ with $\text{supp } f \subset \overline{Q_i^{n*}}$ and $\mathcal{F}^{-1}f \in L_v^p(\mathbb{R}^d)$,*

*then $\mathcal{F}^{-1}(\psi f) = (\mathcal{F}^{-1}\psi) * (\mathcal{F}^{-1}f) \in L_v^p(\mathbb{R}^d)$ with*

$$\|\mathcal{F}^{-1}(\psi f)\|_{L_v^p} \leq [4R_{\mathcal{Q}} \cdot (3C_{\mathcal{Q}})^n]^{d(\frac{1}{p}-1)} \cdot |\det T_i|^{\frac{1}{p}-1} \cdot \|\mathcal{F}^{-1}\psi\|_{L_{v_0}^p} \cdot \|\mathcal{F}^{-1}f\|_{L_v^p}$$

and

$$\|\mathcal{F}^{-1}(\psi f)\|_{W_{T_i^{-T}[-1, 1]^d}(L_v^p)} \leq C \cdot |\det T_i|^{\frac{1}{p}-1} \cdot \|\mathcal{F}^{-1}\psi\|_{L_{v_0}^p} \cdot \|\mathcal{F}^{-1}f\|_{L_v^p},$$

where $C := \Omega_0^K \Omega_1 \cdot \left(2^{14} \cdot d^{\frac{3}{2}} \cdot \left\lceil K + \frac{d+1}{p} \right\rceil\right)^{K+\frac{d+1}{p}+2} [1+4R_{\mathcal{Q}}(3C_{\mathcal{Q}})^n]^{d(\frac{2}{p}-1)}$. ◀

Remark. • Again, the only property of v which we use is that v is measurable and $v(x+y) \leq v(x)v_0(y)$ for all $x, y \in \mathbb{R}^d$. Since this also holds for v_0 instead of v , the claim also holds with v replaced by v_0 everywhere.

- Since $\overline{Q_j} \subset \mathcal{O}$ is compact for each $j \in I$, the same is true of $\overline{Q_i^{n*}} \subset \mathcal{O}$. Hence, the distribution $f \in \mathcal{D}'(\mathcal{O})$ extends to a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$, so that $\mathcal{F}^{-1}f$ is well-defined and such that $\psi f \in \mathcal{S}'(\mathbb{R}^d)$ is a tempered distribution with compact support, since $\psi \in C_c^\infty(\mathbb{R}^d)$. Finally, it follows from [41, Proposition 2.3.22(11)] that $\mathcal{F}^{-1}(\psi f) = \mathcal{F}^{-1}\psi * \mathcal{F}^{-1}f$. ♦

Proof. First, we note that [77, Lemma 2.7] yields

$$Q_j \subset T_i \left[\overline{B_{(1+2C_Q)^n R_Q}}(0) \right] + b_i \quad \forall j \in i^{n*}.$$

Hence, setting $R := (1 + 2C_Q)^n R_Q$, we have

$$\overline{Q_i^{n*}} \subset T_i \overline{B_R}(0) + b_i \subset T_i [-R, R]^d + b_i =: \Omega. \quad (2.19)$$

Note that, once we have proved the first claimed estimate, the second one is a consequence of Theorem 2.17 (and some simple estimates of the resulting constant, using $C_Q \geq \|T_i^{-1} T_i\| = 1$ and $s_d \leq 2^{2d}$), since we have $\text{supp } \mathcal{F}[\mathcal{F}^{-1}(\psi f)] \subset \text{supp } \psi \subset \overline{Q_i^{n*}} \subset \Omega$.

As seen in the remark following the proposition, we have $\mathcal{F}^{-1} f \in \mathcal{S}'(\mathbb{R}^d)$ with $\text{supp } \mathcal{F}[\mathcal{F}^{-1} f] \subset \overline{Q_i^{n*}} \subset \Omega$ and likewise $\mathcal{F}^{-1} \psi \in \mathcal{S}(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$ with $\text{supp } \mathcal{F}[\mathcal{F}^{-1} \psi] \subset \overline{Q_i^{n*}} \subset \Omega$. In view of Theorems 2.17 and 2.19, we thus get $\mathcal{F}^{-1} \psi \in W_{T_i^{-T}[-1,1]^d}(L_{v_0}^p) \hookrightarrow L_{v_0}^1(\mathbb{R}^d)$ (cf. Lemmas 2.3 and 2.7) and $\mathcal{F}^{-1} f \in W_{T_i^{-T}[-1,1]^d}(L_v^p) \hookrightarrow L_v^\infty(\mathbb{R}^d)$, so that $\mathcal{F}^{-1} \psi * \mathcal{F}^{-1} f$ is pointwise well-defined by Corollary 2.22.

Now, Lemma 2.16 ensures existence of a sequence $(h_n)_{n \in \mathbb{N}}$ of Schwartz functions satisfying $|h_n(x)| \leq |(\mathcal{F}^{-1} f)(x)|$, as well as $h_n(x) \xrightarrow{n \rightarrow \infty} (\mathcal{F}^{-1} f)(x)$ for all $x \in \mathbb{R}^d$ and finally $\text{supp } \widehat{h_n} \subset B_{1/n}(\Omega)$ for all $n \in \mathbb{N}$. It is not hard to see $\Omega - B_{1/n}(\Omega) \subset B_{1/n}(\Omega - \Omega)$. Furthermore, by compactness of $\Omega - \Omega$ —and using continuity of the Lebesgue measure from above, cf. [29, Theorem 1.8(d)]—we get

$$\begin{aligned} \lambda_d(B_{1/n}(\Omega - \Omega)) &\xrightarrow{n \rightarrow \infty} \lambda_d(\Omega - \Omega) = \lambda_d\left([T_i[-R, R]^d + b_i] - [T_i[-R, R]^d + b_i]\right) \\ &\leq \lambda_d(T_i[-2R, 2R]^d) = (4R)^d \cdot |\det T_i|. \end{aligned}$$

Now, since $h_n, \mathcal{F}^{-1} \psi \in \mathcal{S}(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$, Theorem 2.15 yields

$$\begin{aligned} v(x) \cdot (|\mathcal{F}^{-1} \psi| * |h_n|)(x) &\leq \left[\lambda_d(\text{supp } \mathcal{F}[\mathcal{F}^{-1} \psi] - \text{supp } \widehat{h_n}) \right]^{\frac{1}{p}-1} \cdot \left[\int_{\mathbb{R}^d} [v(x)]^p \cdot |\mathcal{F}^{-1} \psi(x-y)|^p \cdot |h_n(y)|^p dy \right]^{1/p} \\ (\text{since } v(x) \leq v_0(x-y)v(y)) &\leq [\lambda_d(B_{1/n}(\Omega - \Omega))]^{\frac{1}{p}-1} \cdot \left[\int_{\mathbb{R}^d} |(v_0 \cdot \mathcal{F}^{-1} \psi)(x-y)|^p \cdot |(v \cdot h_n)(y)|^p dy \right]^{1/p} \\ (\text{since } |h_n| \leq |\mathcal{F}^{-1} f|) &\leq [\lambda_d(B_{1/n}(\Omega - \Omega))]^{\frac{1}{p}-1} \cdot \left[(|v_0 \cdot \mathcal{F}^{-1} \psi|^p * |v \cdot \mathcal{F}^{-1} f|^p)(x) \right]^{1/p}. \end{aligned}$$

Taking the limes inferior on both sides, we get

$$\liminf_{n \rightarrow \infty} [v(x) \cdot (|\mathcal{F}^{-1} \psi| * |h_n|)(x)] \leq [(4R)^d \cdot |\det T_i|]^{\frac{1}{p}-1} \cdot \left[(|v_0 \cdot \mathcal{F}^{-1} \psi|^p * |v \cdot \mathcal{F}^{-1} f|^p)(x) \right]^{1/p}.$$

Next, since $h_n \rightarrow \mathcal{F}^{-1} f$ pointwise, and since we saw above that $\mathcal{F}^{-1} \psi * \mathcal{F}^{-1} f$ is pointwise well-defined, we get by Fatou's Lemma that

$$\begin{aligned} |(\mathcal{F}^{-1} \psi * \mathcal{F}^{-1} f)(x)| &\leq \int_{\mathbb{R}^d} |(\mathcal{F}^{-1} \psi)(y)| \cdot |(\mathcal{F}^{-1} f)(x-y)| dy \\ &= \int_{\mathbb{R}^d} \liminf_{n \rightarrow \infty} [|(\mathcal{F}^{-1} \psi)(y)| \cdot |h_n(x-y)|] dy \\ &\leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^d} |(\mathcal{F}^{-1} \psi)(y)| \cdot |h_n(x-y)| dy = \liminf_{n \rightarrow \infty} (|\mathcal{F}^{-1} \psi| * |h_n|)(x) \end{aligned}$$

for all $x \in \mathbb{R}^d$. Hence, we finally see

$$\begin{aligned} \|\mathcal{F}^{-1}(\psi f)\|_{L_v^p} &= \|\mathcal{F}^{-1} \psi * \mathcal{F}^{-1} f\|_{L_v^p} \leq \left\| x \mapsto \liminf_{n \rightarrow \infty} v(x) \cdot (|\mathcal{F}^{-1} \psi| * |h_n|)(x) \right\|_{L_v^p} \\ &\leq [(4R)^d \cdot |\det T_i|]^{\frac{1}{p}-1} \cdot \left\| x \mapsto \left[(|v_0 \cdot \mathcal{F}^{-1} \psi|^p * |v \cdot \mathcal{F}^{-1} f|^p)(x) \right]^{1/p} \right\|_{L^p} \\ &= [(4R)^d \cdot |\det T_i|]^{\frac{1}{p}-1} \cdot \left\| |v_0 \cdot \mathcal{F}^{-1} \psi|^p * |v \cdot \mathcal{F}^{-1} f|^p \right\|_{L^1}^{1/p} \\ (\text{Young's inequality}) &\leq [(4R)^d \cdot |\det T_i|]^{\frac{1}{p}-1} \cdot \left\| |v_0 \cdot \mathcal{F}^{-1} \psi|^p \right\|_{L^1}^{1/p} \cdot \left\| |v \cdot \mathcal{F}^{-1} f|^p \right\|_{L^1}^{1/p} \\ &= [(4R)^d \cdot |\det T_i|]^{\frac{1}{p}-1} \cdot \|\mathcal{F}^{-1} \psi\|_{L_{v_0}^p} \cdot \|\mathcal{F}^{-1} f\|_{L_v^p} < \infty. \end{aligned}$$

Since we have $C_Q \geq \|T_i^{-1}T_i\| = 1$, we get $R = (1 + 2C_Q)^n R_Q \leq (3C_Q)^n R_Q$, which easily yields the claim. \square

As our last result in this section, we show—as a consequence of our developed convolution relations—that the decomposition space $\mathcal{D}(Q, L_v^p, \ell_w^q)$ is well-defined, even if $v \neq 1$.

Proposition 2.24. *Let $\Phi = (\varphi_i)_{i \in I}$ and $\Psi = (\psi_i)_{i \in I}$ be two Q - v_0 -BAPUs. Then we have*

$$\left\| \left(\|\mathcal{F}^{-1}(\varphi_i f)\|_{L_v^p} \right)_{i \in I} \right\|_{\ell_w^q} \asymp \left\| \left(\|\mathcal{F}^{-1}(\psi_i f)\|_{L_v^p} \right)_{i \in I} \right\|_{\ell_w^q}$$

uniformly over $f \in \mathcal{D}'(\mathcal{O})$. In particular, the decomposition space $\mathcal{D}(Q, L_v^p, \ell_w^q)$ is independent of the choice of the Q - v_0 -BAPU. \blacktriangleleft

Proof. By symmetry, it suffices to establish the estimate “ \lesssim ”. We can clearly assume $\left\| \left(\|\mathcal{F}^{-1}(\psi_i f)\|_{L_v^p} \right)_{i \in I} \right\|_{\ell_w^q} < \infty$.

Since $L_v^p(\mathbb{R}^d)$ is a quasi-normed space and since we have the uniform estimate $|i^*| \leq N_Q$ for all $i \in I$, we have

$$d_i := \|\mathcal{F}^{-1}(\psi_i^* f)\|_{L_v^p} \leq C \cdot \sum_{\ell \in i^*} \|\mathcal{F}^{-1}(\psi_\ell f)\|_{L_v^p} = C \cdot (\Gamma_Q e)_i,$$

for a suitable constant $C = C(p, N_Q)$, where $e = (e_i)_{i \in I}$ is defined by $e_i := \|\mathcal{F}^{-1}(\psi_i f)\|_{L_v^p}$ and where Γ_Q is the Q -clustering map, as defined in Section 1.3, equation (1.14).

Now, as seen in Section 1.3, we have $\psi_i^* \equiv 1$ on Q_i and thus $\varphi_i = \psi_i^* \varphi_i$ for all $i \in I$. Hence,

$$c_i := \|\mathcal{F}^{-1}(\varphi_i f)\|_{L_v^p} = \|\mathcal{F}^{-1}(\varphi_i \psi_i^* f)\|_{L_v^p} = \|[\mathcal{F}^{-1} \varphi_i] * \mathcal{F}^{-1}(\psi_i^* f)\|_{L_v^p}.$$

In case of $p \in [1, \infty]$, we can now use the weighted Young inequality (equation (1.12)) to derive

$$c_i \leq \|\mathcal{F}^{-1} \varphi_i\|_{L_{v_0}^1} \cdot \|\mathcal{F}^{-1}(\psi_i^* f)\|_{L_v^p} \leq C \cdot C_{Q, \Phi, v_0, p} \cdot (\Gamma_Q e)_i.$$

Otherwise, if $p \in (0, 1)$, we use Proposition 2.23 (with $n = 1$, since $\text{supp } \varphi_i \subset \overline{Q_i^*}$ and $\text{supp } \psi_i^* \subset \overline{Q_i^*}$) to derive

$$\begin{aligned} c_i &= \|\mathcal{F}^{-1}(\varphi_i \psi_i^* f)\|_{L_v^p} \leq [12R_Q C_Q]^{d(\frac{1}{p}-1)} \cdot |\det T_i|^{\frac{1}{p}-1} \cdot \|\mathcal{F}^{-1} \varphi_i\|_{L_{v_0}^p} \cdot \|\mathcal{F}^{-1}(\psi_i^* f)\|_{L_v^p} \\ &\leq C \cdot [12R_Q C_Q]^{d(\frac{1}{p}-1)} C_{Q, \Phi, v_0, p} \cdot (\Gamma_Q e)_i. \end{aligned}$$

In summary, there is for arbitrary $p \in (0, \infty]$ a constant $C' = C'(Q, p, \Phi, v_0) > 0$ satisfying $c_i \leq C' \cdot (\Gamma_Q e)_i < \infty$ for all $i \in I$. By solidity of $\ell_w^q(I)$ and by boundedness of Γ_Q , this implies

$$\left\| \left(\|\mathcal{F}^{-1}(\varphi_i f)\|_{L_v^p} \right)_{i \in I} \right\|_{\ell_w^q} \leq C' \cdot \|\Gamma_Q e\|_{\ell_w^q} \leq C' \cdot \|\Gamma_Q\| \cdot \|e\|_{\ell_w^q} = C' \cdot \|\Gamma_Q\| \cdot \left\| \left(\|\mathcal{F}^{-1}(\psi_i f)\|_{L_v^p} \right)_{i \in I} \right\|_{\ell_w^q}. \quad \square$$

3. SEMI-DISCRETE BANACH FRAMES

Assumption 3.1. In the remainder of the paper, we will use the following assumptions and notations:

- (1) We are given a family $\Gamma = (\gamma_i)_{i \in I}$ of functions $\gamma_i : \mathbb{R}^d \rightarrow \mathbb{C}$ with the following additional properties:
 - (a) We have $\gamma_i \in L_{(1+|\bullet|)^\kappa}^1(\mathbb{R}^d) \hookrightarrow L_{v_0}^1(\mathbb{R}^d) \hookrightarrow L^1(\mathbb{R}^d)$ for all $i \in I$.
 - (b) We have $\widehat{\gamma}_i \in C^\infty(\mathbb{R}^d)$ for all $i \in I$, where all partial derivatives of $\widehat{\gamma}_i$ are polynomially bounded, i.e.,

$$|(\partial^\alpha \widehat{\gamma}_i)(\xi)| \leq C_{\alpha, i} \cdot (1 + |\xi|)^{N_{\alpha, i}} \quad \forall \xi \in \mathbb{R}^d \forall \alpha \in \mathbb{N}_0^d \forall i \in I, \text{ for suitable } C_{\alpha, i} > 0 \text{ and } N_{\alpha, i} \in \mathbb{N}_0.$$

- (2) For $i \in I$, we define

$$\begin{aligned} \gamma^{(i)} &:= \mathcal{F}^{-1}(\widehat{\gamma}_i \circ S_i^{-1}) \\ &= \mathcal{F}^{-1}(L_{b_i}(\widehat{\gamma}_i \circ T_i^{-1})) \\ &= M_{b_i}[\mathcal{F}^{-1}(\widehat{\gamma}_i \circ T_i^{-1})] \\ &= |\det T_i| \cdot M_{b_i}[\gamma_i \circ T_i^T], \end{aligned} \tag{3.1}$$

as well as the L^2 -normalized version

$$\gamma^{[i]} := |\det T_i|^{1/2} \cdot M_{b_i}[\gamma_i \circ T_i^T] = |\det T_i|^{-1/2} \cdot \gamma^{(i)}. \tag{3.2}$$

(3) For $i \in I$, we set

$$V_i := \begin{cases} L_v^p(\mathbb{R}^d), & \text{if } p \in [1, \infty], \\ W_{T_i^{-T}[-1,1]^d}(L_v^p), & \text{if } p \in (0, 1). \end{cases}$$

Furthermore, we will occasionally make use of the space

$$V := \ell_w^q([V_i]_{i \in I}) := \left\{ (f_i)_{i \in I} \mid (\forall i \in I : f_i \in V_i) \text{ and } (\|f_i\|_{V_i})_{i \in I} \in \ell_w^q(I) \right\},$$

equipped with the quasi-norm $\|(f_i)_{i \in I}\|_{\ell_w^q([V_i]_{i \in I})} := \|(\|f_i\|_{V_i})_{i \in I}\|_{\ell_w^q}$.

(4) Finally, we set

$$r := \max \left\{ q, \frac{q}{p} \right\} = \begin{cases} q, & \text{if } p \in [1, \infty], \\ \frac{q}{p}, & \text{if } p \in (0, 1) \end{cases}$$

and

$$A_{j,i} := \begin{cases} \left\| \mathcal{F}^{-1}(\varphi_i \cdot \widehat{\gamma^{(j)}}) \right\|_{L_{v_0}^1}, & \text{if } p \in [1, \infty], \\ (1 + \|T_j^{-1}T_i\|)^d \cdot |\det T_i|^{1-p} \cdot \left\| \mathcal{F}^{-1}(\varphi_i \cdot \widehat{\gamma^{(j)}}) \right\|_{L_{v_0}^p}^p, & \text{if } p \in (0, 1) \end{cases}$$

for $i, j \in I$ and we assume that \vec{A} is a bounded operator $\vec{A} : \ell_{w^{\min\{1,p\}}}^r(I) \rightarrow \ell_{w^{\min\{1,p\}}}^r(I)$, where

$$\vec{A}(c_i)_{i \in I} := \left(\sum_{i \in I} A_{j,i} c_i \right)_{j \in I}.$$

◀

Remark 3.2. (1) The most common case will be to have $\gamma_i = \gamma$ for all $i \in I$, for a fixed prototype γ . The added flexibility of allowing γ_i to vary with $i \in I$ is only rarely needed. In the cases where it is, we usually have $\gamma_i = \gamma_{n_i}$ with a given (finite) list of prototypes $\gamma_1, \dots, \gamma_N$.

(2) The assumption that $\partial^\alpha \widehat{\gamma_i}$ is polynomially bounded for all $\alpha \in \mathbb{N}_0^d$ is satisfied if $\gamma_i \in L^1(\mathbb{R}^d)$ has compact support, say $\text{supp } \gamma_i \subset [-R, R]^d$ with $R \geq 1$, since then differentiation under the integral yields

$$\begin{aligned} |\partial^\alpha \widehat{\gamma_i}(\xi)| &= \left| \int_{\mathbb{R}^d} \gamma_i(x) \cdot \partial_\xi^\alpha e^{-2\pi i \langle x, \xi \rangle} dx \right| \\ &\leq \int_{[-R, R]^d} |\gamma_i(x)| \cdot (2\pi|x|)^{|\alpha|} dx \leq (2\pi R)^{|\alpha|} \cdot \|\gamma_i\|_{L^1} < \infty \end{aligned}$$

for all $\xi \in \mathbb{R}^d$ and arbitrary $\alpha \in \mathbb{N}_0^d$.

(3) Under the above assumptions, the chain rule implies

$$\begin{aligned} \left| \left(\partial^\alpha \widehat{\gamma^{(i)}} \right)(x) \right| &= \left| \left(\partial^\alpha [\widehat{\gamma_i} \circ T_i^{-1}] \right)(x - b_i) \right| \\ &\leq C^{(\alpha)} \cdot \|T_i^{-1}\|^{|\alpha|} \cdot \max_{|\beta| \leq |\alpha|} \left| \left(\partial^\beta \widehat{\gamma_i} \right)(T_i^{-1}(x - b_i)) \right| \\ &\leq \left(\max_{|\beta| \leq |\alpha|} C_{\beta,i} \right) \cdot C^{(\alpha)} \cdot \|T_i^{-1}\|^{|\alpha|} \cdot \max_{|\beta| \leq |\alpha|} (1 + |T_i^{-1}(x - b_i)|)^{N_{\beta,i}} \\ &\quad (\text{with } N'_{\alpha,i} := \max_{|\beta| \leq |\alpha|} N_{\beta,i}) \leq C'_{\alpha,i} \cdot (1 + |T_i^{-1}(x - b_i)|)^{N'_{\alpha,i}} \\ &\quad (\text{for suitable } C''_{\alpha,i} > 0) \leq C''_{\alpha,i} \cdot (1 + |x|)^{N'_{\alpha,i}}, \end{aligned}$$

where the last step used

$$\begin{aligned} 1 + |T_i^{-1}(x - b_i)| &\leq 1 + \|T_i^{-1}\| |x - b_i| \\ &\leq 1 + \|T_i^{-1}\| |b_i| + \|T_i^{-1}\| |x| \\ &\leq (1 + \|T_i^{-1}\| |b_i| + \|T_i^{-1}\|) \cdot (1 + |x|). \end{aligned}$$

Hence, all partial derivatives of each $\widehat{\gamma^{(i)}}$ are polynomially bounded.

(4) Using $\gamma_i \in L^1_{(1+|\bullet|)^K}(\mathbb{R}^d)$, we also get

$$\begin{aligned} \left\| \gamma^{(i)} \right\|_{L^1_{(1+|\bullet|)^K}} &= |\det T_i| \cdot \left\| (1+|\bullet|)^K \cdot (\gamma_i \circ T_i^T) \right\|_{L^1} \\ &= \left\| (1+|T_i^{-T}\bullet|)^K \cdot \gamma_i \right\|_{L^1} \\ &\stackrel{(\text{eq. (1.11)})}{\leq} \Omega_0^K \cdot \left\| (1+|\bullet|)^K \cdot \gamma_i \right\|_{L^1} = \Omega_0^K \cdot \|\gamma_i\|_{L^1_{(1+|\bullet|)^K}} < \infty \end{aligned}$$

and thus $\gamma^{(i)} \in L^1_{(1+|\bullet|)^K}(\mathbb{R}^d) \hookrightarrow L^1_{v_0}(\mathbb{R}^d) \hookrightarrow L^1(\mathbb{R}^d)$, where the last embedding uses $v_0 \geq 1$.

(5) Point (3) from above shows $\widehat{\gamma^{(i)}} \cdot \widehat{f} \in \mathcal{S}'(\mathbb{R}^d)$ for arbitrary $f \in \mathcal{S}'(\mathbb{R}^d)$, so that $\gamma^{(i)} * f := \mathcal{F}^{-1}(\widehat{\gamma^{(i)}} \cdot \widehat{f})$ is a well-defined tempered distribution. Of course, the same also holds for $\gamma^{[i]} * f := \mathcal{F}^{-1}(\widehat{\gamma^{[i]}} \cdot \widehat{f})$.

(6) Since \mathbb{R}^d is σ -compact, it follows from [76, Lemma 2.3.7] (see also [68, Theorem 2.6]) that for $p \in (0, 1)$, each of the spaces $V_i = W_{T_i^{-T}[-1, 1]^d}(L_v^p)$ is complete (and thus a Quasi-Banach space) for each $i \in I$. Furthermore, [76, Lemma 2.3.4] and [41, Exercise 1.1.5(c)] show $\|f + g\|_{V_i} \leq 2^{\frac{1}{p}-1} \cdot [\|f\|_{V_i} + \|g\|_{V_i}]$ for all $f, g \in V_i$. In case of $p \in [1, \infty]$, it is clear that $V_i = L_v^p(\mathbb{R}^d)$ is a Banach space. \blacklozenge

Note that in the preceding remark, we defined $\gamma^{(i)} * f := \mathcal{F}^{-1}(\widehat{\gamma^{(i)}} \cdot \widehat{f})$. We needed to do so, since the usual results about convolution in $\mathcal{S}'(\mathbb{R}^d)$ only define $f * \varphi$ for $\varphi \in \mathcal{S}'(\mathbb{R}^d)$ if $f \in \mathcal{S}(\mathbb{R}^d)$ (cf. [29, Proposition (8.44)]) or if f is a distribution with compact support. (cf. [29, Chapter 8, Exercise 35]). But note that if we not only know $\varphi \in \mathcal{S}'(\mathbb{R}^d)$, but the stronger property $\varphi \in (L^1 + L^\infty)(\mathbb{R}^d)$ and if $f \in L^1(\mathbb{R}^d)$, then $f * \varphi \in (L^1 + L^\infty)(\mathbb{R}^d)$ is already defined. Our next result shows that in this (and in a slightly more general) case, the new definition is consistent.

Lemma 3.3. *Assume $\varphi \in L_v^1(\mathbb{R}^d) + L_v^\infty(\mathbb{R}^d)$ and assume that $f \in L_{v_0}^1(\mathbb{R}^d)$ is such that $\widehat{f} \in C^\infty(\mathbb{R}^d)$ and such that all partial derivatives of \widehat{f} have at most polynomial growth. Then $f * \varphi \in L_v^1(\mathbb{R}^d) + L_v^\infty(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$ and*

$$f * \varphi = \mathcal{F}^{-1}[\widehat{f} \cdot \widehat{\varphi}].$$

The assumption on φ is in particular fulfilled if $\varphi \in V_i$ for some $i \in I$. More precisely,

$$V_i \hookrightarrow L_v^1(\mathbb{R}^d) + L_v^\infty(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d). \quad (3.3)$$

Remark 3.4. We saw in the proof of Theorem 2.19 (cf. equation (2.15)) that $v(x) \gtrsim (1+|x|)^{-K}$. Furthermore, $v_0 \geq 1$, and thus $L_v^p(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$ and $L_{v_0}^p(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$ for all $p \in [1, \infty]$. Hence, the expressions \widehat{f} and $\widehat{\varphi}$ above are well-defined tempered distributions.

Since $\widehat{f} \in C^\infty(\mathbb{R}^d)$ with all derivatives of \widehat{f} of at most polynomial growth, we also see $\widehat{f} \cdot \widehat{\varphi} \in \mathcal{S}'(\mathbb{R}^d)$ and thus also $\mathcal{F}^{-1}[\widehat{f} \cdot \widehat{\varphi}] \in \mathcal{S}'(\mathbb{R}^d)$. \blacklozenge

Proof. From the weighted Young inequality (equation (1.12)), we know $L_{v_0}^1(\mathbb{R}^d) * L_v^\infty(\mathbb{R}^d) \hookrightarrow L_v^\infty(\mathbb{R}^d)$. Likewise, the same inequality also yields $\|f * g\|_{L_v^1} \leq \|f\|_{L_{v_0}^1} \cdot \|g\|_{L_v^1} < \infty$ and thus in particular $(|f| * |g|)(x) < \infty$ for almost all $x \in \mathbb{R}^d$ for $f \in L_{v_0}^1(\mathbb{R}^d)$ and $g \in L_v^1(\mathbb{R}^d)$. Hence, together with Remark 3.4, we see indeed that $f * \varphi \in L_v^1(\mathbb{R}^d) + L_v^\infty(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$.

Now, let $\psi \in \mathcal{S}(\mathbb{R}^d)$ be arbitrary. We have

$$\begin{aligned} \left\langle \mathcal{F}^{-1}[\widehat{f} \cdot \widehat{\varphi}], \psi \right\rangle_{\mathcal{S}', \mathcal{S}} &= \left\langle \widehat{f} \cdot \widehat{\varphi}, \mathcal{F}^{-1}\psi \right\rangle_{\mathcal{S}', \mathcal{S}} \\ &= \left\langle \widehat{\varphi}, \widehat{f} \cdot \mathcal{F}^{-1}\psi \right\rangle_{\mathcal{S}', \mathcal{S}} \\ &= \left\langle \varphi, \mathcal{F}[\widehat{f} \cdot \mathcal{F}^{-1}\psi] \right\rangle_{\mathcal{S}', \mathcal{S}} \\ &= \left\langle \varphi, \left(\mathcal{F}^{-1}[\widehat{f} \cdot \mathcal{F}^{-1}\psi] \right)(-\bullet) \right\rangle_{\mathcal{S}', \mathcal{S}}. \end{aligned}$$

Recall that $v_0 \geq 1$, so that $f \in L_{v_0}^1(\mathbb{R}^d) \hookrightarrow L^1(\mathbb{R}^d)$. Hence, $h := f * \tilde{\psi} \in L^1(\mathbb{R}^d) * L^1(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$, where $\tilde{\psi}(x) := \psi(-x)$. Thus, the convolution theorem yields $\widehat{h} = \widehat{f} \cdot \widehat{\tilde{\psi}} = \widehat{f} \cdot \widehat{\psi} = \widehat{f} \cdot \mathcal{F}^{-1}\psi \in \mathcal{S}(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$, since

$\mathcal{F}^{-1}\psi \in \mathcal{S}(\mathbb{R}^d)$ and since all partial derivatives of \widehat{f} are polynomially bounded. By the Fourier inversion theorem, this implies $f * \tilde{\psi} = h = \mathcal{F}^{-1}\widehat{h} = \mathcal{F}^{-1}[\widehat{f} \cdot \mathcal{F}^{-1}\psi]$, so that we can continue the calculation from above as follows:

$$\begin{aligned} \langle \mathcal{F}^{-1}[\widehat{f} \cdot \widehat{\varphi}], \psi \rangle_{\mathcal{S}', \mathcal{S}} &= \langle \varphi, (f * \tilde{\psi})(-\bullet) \rangle_{\mathcal{S}', \mathcal{S}} \\ &= \int_{\mathbb{R}^d} \varphi(x) \cdot \int_{\mathbb{R}^d} f(-x-y) \cdot \tilde{\psi}(y) \, dy \, dx \\ (z=-y) &= \int_{\mathbb{R}^d} \varphi(x) \cdot \int_{\mathbb{R}^d} f(z-x) \cdot \psi(z) \, dz \, dx \\ (\text{Fubini}) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(x) \cdot f(z-x) \, dx \cdot \psi(z) \, dz \\ &= \langle f * \varphi, \psi \rangle_{\mathcal{S}', \mathcal{S}}, \end{aligned}$$

which proves the claim.

All that remains is to justify the application of Fubini's theorem. To this end, we can assume $\varphi \in L_v^1(\mathbb{R}^d)$ or $\varphi \in L_v^\infty(\mathbb{R}^d)$, since then the general case follows by linearity. But for $\varphi \in L_v^\infty(\mathbb{R}^d)$, we have because of

$$\begin{aligned} v(0) &= v(x + (-x)) \\ &\leq v(x) \cdot v_0(-x) \\ &= v(x) \cdot v_0(z - x + (-z)) \\ &\leq v(x) \cdot v_0(z - x) \cdot v_0(-z) \\ &\leq \Omega_1 \cdot v(x) \cdot v_0(z - x) \cdot (1 + |z|)^K \end{aligned}$$

that

$$\begin{aligned} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\varphi(x) \cdot f(z-x) \cdot \psi(z)| \, dx \, dz &\leq \frac{\Omega_1}{v(0)} \cdot \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |(v \cdot \varphi)(x)| \cdot |(v_0 \cdot f)(z-x)| \cdot (1 + |z|)^K |\psi(z)| \, dx \, dz \\ &\leq \frac{\Omega_1}{v(0)} \cdot \|\varphi\|_{L_v^\infty} \cdot \int_{\mathbb{R}^d} (1 + |z|)^K |\psi(z)| \, dz \int_{\mathbb{R}^d} |(v_0 \cdot f)(z-x)| \, dx \, dz \\ (y=z-x) &= \frac{\Omega_1}{v(0)} \cdot \|\varphi\|_{L_v^\infty} \|f\|_{L_{v_0}^1} \int_{\mathbb{R}^d} (1 + |z|)^K |\psi(z)| \, dz < \infty. \end{aligned}$$

Furthermore, in case of $\varphi \in L_v^1(\mathbb{R}^d)$, we get with a similar estimate that

$$\begin{aligned} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\varphi(x) \cdot f(z-x) \cdot \psi(z)| \, dx \, dz &\leq \frac{\Omega_1}{v(0)} \cdot \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |(v \cdot \varphi)(x)| \cdot |(v_0 \cdot f)(z-x)| \cdot (1 + |z|)^K |\psi(z)| \, dx \, dz \\ &\leq \frac{\Omega_1}{v(0)} \cdot \left[\sup_{z \in \mathbb{R}^d} (1 + |z|)^K |\psi(z)| \right] \cdot \int_{\mathbb{R}^d} |(v \cdot \varphi)(x)| \, dx \int_{\mathbb{R}^d} |(v_0 \cdot f)(z-x)| \, dz \, dx \\ &= \frac{\Omega_1}{v(0)} \cdot \left[\sup_{z \in \mathbb{R}^d} (1 + |z|)^K |\psi(z)| \right] \cdot \|\varphi\|_{L_v^1} \cdot \|f\|_{L_{v_0}^1} < \infty. \end{aligned}$$

For the proof of equation (3.3), note for $p \in [1, \infty]$ that $V_i = L_v^p(\mathbb{R}^d) \hookrightarrow L_v^1(\mathbb{R}^d) + L_v^\infty(\mathbb{R}^d)$, because of the well-known (cf. [29, Proposition (6.9)]) embedding $L^p(\mathbb{R}^d) \hookrightarrow L^1(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$. But in case of $p \in (0, 1)$, Theorem 2.19 and the ensuing remark yield $V_i = W_{T_i^{-x}[-1,1]^d}(L_v^p) \hookrightarrow L_v^\infty(\mathbb{R}^d) \hookrightarrow L_v^1(\mathbb{R}^d) + L_v^\infty(\mathbb{R}^d)$, as desired. \square

One of our aims in this section is to show under the conditions of Assumption 3.1 (and certain additional assumptions, cf. Assumption 3.6) on $\mathcal{Q}, \Gamma = (\gamma_i)_{i \in I}$ and p, q, v, w that we have

$$\|f\|_{\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)} \asymp \left\| \left(\|\gamma^{(i)} * f\|_{V_i} \right)_{i \in I} \right\|_{\ell_w^q} = \left\| \left(\gamma^{(i)} * f \right)_{i \in I} \right\|_{V} \quad (3.4)$$

for all $f \in \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$. Note though, that it is not a priori clear how the convolution $\gamma^{(i)} * f$ can be interpreted, since we have $f \in \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q) \leq \mathcal{F}^{-1}(\mathcal{D}'(\mathcal{O})) \not\subseteq \mathcal{S}'(\mathbb{R}^d)$. The purpose of the following result is to clarify how $\gamma^{(i)} * f$ can be interpreted and to establish the estimate “ \gtrsim ” in equation (3.4). We remark that the theorem uses the notion of **normal convergence** of a series. In our context, we say that a series $\sum_{i \in I} g_i$ converges normally in

V_j if

$$\begin{cases} \sum_{i \in I} \|g_i\|_{V_j} < \infty, & \text{if } p \in [1, \infty], \\ \sum_{i \in I} \|g_i\|_{V_j}^p < \infty, & \text{if } p \in (0, 1). \end{cases}$$

Theorem 3.5. If Assumption 3.1 is fulfilled, the following hold:

For every $f \in \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$ and $j \in I$, the distribution $\widehat{\gamma^{(j)}} \cdot \widehat{f} \in \mathcal{D}'(\mathcal{O})$ extends to a tempered distribution $f_j \in \mathcal{F}(V_j) \subset \mathcal{S}'(\mathbb{R}^d)$, given by

$$f_j : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathbb{C}, \phi \mapsto \sum_{i \in I} \left\langle \widehat{\gamma^{(j)}} \cdot \widehat{f}, \varphi_i \phi \right\rangle_{\mathcal{D}'(\mathcal{O}), C_c^\infty(\mathcal{O})}.$$

Furthermore, the inverse Fourier transform $\mathcal{F}^{-1}f_j \in V_j$ is given by

$$(\mathcal{F}^{-1}f_j)(x) = \sum_{i \in I} \left[\mathcal{F}^{-1} \left(\varphi_i \widehat{\gamma^{(j)}} \widehat{f} \right) \right](x), \quad (3.5)$$

where the series converges normally in V_j and absolutely almost everywhere.

Finally, the linear map

$$\text{Ana}_\Gamma : \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q) \rightarrow \ell_w^q([V_i]_{i \in I}), f \mapsto (\mathcal{F}^{-1}f_j)_{j \in I}$$

is well-defined and bounded, with

$$\|\text{Ana}_\Gamma\| \leq C \cdot \|\Gamma_\mathcal{Q}\| \cdot \|\vec{A}\|_{\ell_{w \min\{1, p\}}^r(I) \rightarrow \ell_{w \min\{1, p\}}^r(I)}^{\max\{1, \frac{1}{p}\}},$$

where

$$C := \begin{cases} 1, & \text{if } p \in [1, \infty], \\ N_{\mathcal{Q}}^{\frac{1}{p}-1} \cdot \left(12288 \cdot d^{3/2} \cdot \left\lceil K + \frac{d+1}{p} \right\rceil \right)^{\lceil K + \frac{d+1}{p} \rceil + 1} \cdot (1 + R_{\mathcal{Q}})^{d/p} (12R_{\mathcal{Q}}C_{\mathcal{Q}})^{d(\frac{1}{p}-1)} \cdot \Omega_0^K \Omega_1, & \text{if } p \in (0, 1) \end{cases}$$

and where $\Gamma_\mathcal{Q} : \ell_w^q(I) \rightarrow \ell_w^q(I)$ is the \mathcal{Q} -clustering map, i.e., $\Gamma_\mathcal{Q}(c_i)_{i \in I} = (c_i^*)_{i \in I}$, with $c_i^* := \sum_{\ell \in i^*} c_\ell$. \blacktriangleleft

Remark. In the following, we will use the notation $\gamma^{(j)} * f$ instead of $\mathcal{F}^{-1}f_j$, so that we have

$$\text{Ana}_\Gamma f = \left(\gamma^{(j)} * f \right)_{j \in I}.$$

Likewise, because of $\gamma^{[j]} = |\det T_j|^{-1/2} \cdot \gamma^{(j)}$, it is natural to define

$$\gamma^{[j]} * f := |\det T_j|^{-1/2} \cdot \gamma^{(j)} * f.$$

This new notation $\gamma^{(j)} * f$ (and thus also $\gamma^{[j]} * f$) is consistent in the following sense: If we have $\mathcal{O} = \mathbb{R}^d$ and $\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$ (i.e., if every $f \in \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q) \subset \mathcal{Z}'(\mathbb{R}^d) = [\mathcal{F}(C_c^\infty(\mathbb{R}^d))]'$ extends to a tempered distribution $f_{\mathcal{S}}$), then our new definition of the convolution $\gamma^{(j)} * f := \mathcal{F}^{-1}f_j$ agrees with the usual interpretation of $\gamma^{(j)} * f := \mathcal{F}^{-1}(\widehat{\gamma^{(j)}} \cdot \widehat{f})$ for $f \in \mathcal{S}'(\mathbb{R}^d) \supset \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$, as we will see now.

First note $\widehat{f_{\mathcal{S}}}|_{C_c^\infty(\mathbb{R}^d)} = \widehat{f}$, where $\widehat{f} = f \circ \mathcal{F} \in \mathcal{D}'(\mathbb{R}^d)$. Thus, we have for arbitrary $\phi \in \mathcal{F}(C_c^\infty(\mathbb{R}^d))$ that

$$\begin{aligned} \langle \mathcal{F}^{-1}f_j, \phi \rangle_{\mathcal{S}', \mathcal{S}} &= \langle f_j, \mathcal{F}^{-1}\phi \rangle_{\mathcal{S}', \mathcal{S}} \\ &= \sum_{i \in I} \left\langle \widehat{\gamma^{(j)}} \widehat{f}, \varphi_i \cdot \mathcal{F}^{-1}\phi \right\rangle_{\mathcal{D}'(\mathbb{R}^d), C_c^\infty(\mathbb{R}^d)} \\ &\quad (\text{since } \mathcal{F}^{-1}\phi \in C_c^\infty(\mathbb{R}^d) \text{ and } \sum_{i \in I} \varphi_i \equiv 1 \text{ with a locally finite sum}) = \left\langle \widehat{\gamma^{(j)}} \cdot \widehat{f}, \mathcal{F}^{-1}\phi \right\rangle_{\mathcal{D}'(\mathbb{R}^d), C_c^\infty(\mathbb{R}^d)} \\ &= \left\langle \widehat{f}, \widehat{\gamma^{(j)}} \cdot \mathcal{F}^{-1}\phi \right\rangle_{\mathcal{D}'(\mathbb{R}^d), C_c^\infty(\mathbb{R}^d)} \\ &\quad (\widehat{\gamma^{(j)}} \cdot \mathcal{F}^{-1}\phi \in C_c^\infty(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d), \text{ since } \widehat{\gamma^{(j)}} \in C^\infty(\mathbb{R}^d) \text{ and } \mathcal{F}^{-1}\phi \in C_c^\infty(\mathbb{R}^d)) = \left\langle \widehat{f_{\mathcal{S}}}, \widehat{\gamma^{(j)}} \cdot \mathcal{F}^{-1}\phi \right\rangle_{\mathcal{S}', \mathcal{S}} \\ &\quad (\widehat{\gamma^{(j)}} \cdot \widehat{f_{\mathcal{S}}} \in \mathcal{S}'(\mathbb{R}^d), \text{ since } \widehat{f_{\mathcal{S}}} \in \mathcal{S}'(\mathbb{R}^d) \text{ and all derivatives of } \widehat{\gamma^{(j)}} \text{ pol. bounded}) = \left\langle \widehat{\gamma^{(j)}} \cdot \widehat{f_{\mathcal{S}}}, \mathcal{F}^{-1}\phi \right\rangle_{\mathcal{S}', \mathcal{S}} \\ &= \left\langle \mathcal{F}^{-1}[\widehat{\gamma^{(j)}} \cdot \widehat{f_{\mathcal{S}}}], \phi \right\rangle_{\mathcal{S}', \mathcal{S}} = \left\langle \gamma^{(j)} * f_{\mathcal{S}}, \phi \right\rangle_{\mathcal{S}', \mathcal{S}}. \end{aligned}$$

Here, the last step uses the definition $\gamma^{(j)} * f_S := \mathcal{F}^{-1} \left[\widehat{\gamma^{(j)}} \cdot \widehat{f_S} \right]$ from above. This definition coincides with the usual one if $\gamma_i \in \mathcal{S}(\mathbb{R}^d)$ for all $i \in I$ (so that $\gamma^{(j)} \in \mathcal{S}(\mathbb{R}^d)$) or (by Lemma 3.3 and since $\gamma^{(j)} \in L_{v_0}^1(\mathbb{R}^d)$ as seen in Remark 3.2) if $f_S \in (L_v^1 + L_v^\infty)(\mathbb{R}^d)$, which is satisfied in many cases.

Now, since $\mathcal{F}(C_c^\infty(\mathbb{R}^d))$ is dense in $\mathcal{S}(\mathbb{R}^d)$ (cf. [29, Proposition 9.9]) and since we have $\mathcal{F}^{-1}f_j \in V_j \subset \mathcal{S}'(\mathbb{R}^d)$ and $\gamma^{(j)} * f_S = \mathcal{F}^{-1} \left(\widehat{\gamma^{(j)}} \cdot \widehat{f_S} \right) \in \mathcal{S}'(\mathbb{R}^d)$, we conclude $\gamma^{(j)} * f_S = \mathcal{F}^{-1}f_j$, as claimed. \blacklozenge

Proof of Theorem 3.5. Let $f \in \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$ be arbitrary and let $c_i := \left\| \mathcal{F}^{-1}(\varphi_i^* \cdot \widehat{f}) \right\|_{L_v^p}$ for $i \in I$. Using the (quasi)-triangle inequality for $L^p(\mathbb{R}^d)$ and the uniform estimate $|i^*| \leq N_{\mathcal{Q}}$, we obtain a constant $C_1 = C_1(p, \mathcal{Q}) > 0$ satisfying

$$c_i = \left\| \mathcal{F}^{-1}(\varphi_i^* \cdot \widehat{f}) \right\|_{L_v^p} \leq C_1 \cdot \sum_{\ell \in i^*} \left\| \mathcal{F}^{-1}(\varphi_\ell \cdot \widehat{f}) \right\|_{L_v^p} = C_1 \cdot (\Gamma_{\mathcal{Q}} d)_i \quad \text{for } d = (d_i)_{i \in I}, \text{ with } d_i := \left\| \mathcal{F}^{-1}(\varphi_i \cdot \widehat{f}) \right\|_{L_v^p}.$$

In fact, as shown in [41, Exercise 1.1.5(c)], we can choose

$$C_1 = \begin{cases} 1, & \text{if } p \in [1, \infty], \\ N_{\mathcal{Q}}^{\frac{1}{p}-1}, & \text{if } p \in (0, 1). \end{cases}$$

Since $d \in \ell_w^q(I)$ with $\|d\|_{\ell_w^q} = \|f\|_{\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)}$, we get $c \in \ell_w^q(I)$ as well, and $\|c\|_{\ell_w^q} \leq C_1 \cdot \|\Gamma_{\mathcal{Q}}\| \cdot \|f\|_{\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)}$.

Now, we distinguish the two cases $p \in [1, \infty]$ and $p \in (0, 1)$.

In case of $p \in [1, \infty]$, we have $V_j = L_v^p(\mathbb{R}^d)$. Here, the weighted Young inequality (equation (1.12)) yields

$$\begin{aligned} \left\| \mathcal{F}^{-1}(\widehat{\gamma^{(j)}} \cdot \varphi_i \cdot \widehat{f}) \right\|_{L_v^p} &= \left\| \mathcal{F}^{-1}(\widehat{\gamma^{(j)}} \cdot \varphi_i \cdot \varphi_i^* \cdot \widehat{f}) \right\|_{L_v^p} \\ &= \left\| \mathcal{F}^{-1}(\widehat{\gamma^{(j)}} \cdot \varphi_i) * \mathcal{F}^{-1}(\varphi_i^* \cdot \widehat{f}) \right\|_{L_v^p} \\ &\leq \left\| \mathcal{F}^{-1}(\widehat{\gamma^{(j)}} \cdot \varphi_i) \right\|_{L_{v_0}^1} \cdot \left\| \mathcal{F}^{-1}(\varphi_i^* \cdot \widehat{f}) \right\|_{L_v^p} \\ &= A_{j,i} \cdot c_i, \end{aligned}$$

with $A_{j,i}$ as in Assumption 3.1. Hence, we get

$$\sum_{i \in I} \left\| \mathcal{F}^{-1}(\widehat{\gamma^{(j)}} \cdot \varphi_i \cdot \widehat{f}) \right\|_{L_v^p} \leq \sum_{i \in I} [A_{j,i} \cdot c_i] = \left(\vec{A} \cdot c \right)_j < \infty, \quad (3.6)$$

since we have $c \in \ell_w^q(I)$ and since Assumption 3.1 includes (for $p \in [1, \infty]$) the assumption that $\vec{A} : \ell_w^q(I) \rightarrow \ell_w^q(I)$ is well-defined and bounded. This implies that the function

$$F_j := \sum_{i \in I} \mathcal{F}^{-1}(\widehat{\gamma^{(j)}} \cdot \varphi_i \cdot \widehat{f}) \in L_v^p(\mathbb{R}^d) = V_j$$

is well-defined, with normal convergence in V_j and with absolute convergence a.e. of the defining series and such that $\|F_j\|_{L_v^p} \leq \left(\vec{A} \cdot c \right)_j$ for all $j \in I$.

Next, in case of $p \in (0, 1)$, define $e_i := c_i^p$ for $i \in I$ and note $e = (e_i)_{i \in I} \in \ell_{w^p}^{q/p}(I) = \ell_{w^{\min\{1,p\}}}^r(I)$, with

$$\|e\|_{\ell_{w^{\min\{1,p\}}}^r(I)} = \|(w_i^p \cdot c_i^p)_{i \in I}\|_{\ell_{q/p}} = \|(w_i \cdot c_i)_{i \in I}\|_{\ell_q}^p = \|c\|_{\ell_w^q(I)}^p.$$

Next, we note

$$\begin{aligned} \text{supp}(\widehat{\gamma^{(j)}} \cdot \varphi_i \cdot \widehat{f}) &\subset \text{supp} \varphi_i \subset \overline{Q_i} \subset T_i \overline{B_{R_{\mathcal{Q}}}(0)} + b_i \\ &\subset T_j \left[T_j^{-1} T_i \overline{B_{R_{\mathcal{Q}}}(0)} \right] + b_i \\ &\subset T_j \left[\|T_j^{-1} T_i\| \overline{B_{R_{\mathcal{Q}}}(0)} \right] + b_i \\ &\subset T_j \left[-\|T_j^{-1} T_i\| R_{\mathcal{Q}}, \|T_j^{-1} T_i\| R_{\mathcal{Q}} \right]^d + b_i, \end{aligned}$$

so that Theorem 2.17 yields for $C_2 := 2^{4(1+\frac{d}{p})} s_d^{\frac{1}{p}} \left(192 \cdot d^{\frac{3}{2}} \cdot \left\lceil K + \frac{d+1}{p} \right\rceil \right)^{\left\lceil K + \frac{d+1}{p} \right\rceil + 1} \cdot \Omega_0^K \Omega_1$ and $C_3 := C_2 \cdot (1 + R_Q)^{d/p}$ that

$$\begin{aligned}
\left\| \mathcal{F}^{-1} \left(\widehat{\gamma^{(j)}} \cdot \varphi_i \cdot \widehat{f} \right) \right\|_{V_j} &= \left\| \mathcal{F}^{-1} \left(\widehat{\gamma^{(j)}} \cdot \varphi_i \cdot \widehat{f} \right) \right\|_{W_{T_j^{-T}[-1,1]^d}(L_v^p)} \\
&\leq C_2 (1 + \|T_j^{-1} T_i\| R_Q)^{d/p} \cdot \left\| \mathcal{F}^{-1} \left(\widehat{\gamma^{(j)}} \cdot \varphi_i \cdot \widehat{f} \right) \right\|_{L_v^p} \\
(1+ab \leq (1+a)(1+b) \text{ for } a, b \geq 0) &\leq C_3 \cdot (1 + \|T_j^{-1} T_i\|)^{d/p} \cdot \left\| \mathcal{F}^{-1} \left(\widehat{\gamma^{(j)}} \cdot \varphi_i \right) * \mathcal{F}^{-1} \left(\varphi_i^* \cdot \widehat{f} \right) \right\|_{L_v^p} \\
(\text{Prop. 2.23 with } n=1) &\leq C_3 \cdot (12R_Q C_Q)^{d(\frac{1}{p}-1)} \cdot (1 + \|T_j^{-1} T_i\|)^{d/p} |\det T_i|^{\frac{1}{p}-1} \left\| \mathcal{F}^{-1} \left(\varphi_i \widehat{\gamma^{(j)}} \right) \right\|_{L_{v_0}^p} \left\| \mathcal{F}^{-1} \left(\varphi_i^* \widehat{f} \right) \right\|_{L_v^p} \\
&\leq C_3 \cdot (12R_Q C_Q)^{d(\frac{1}{p}-1)} \cdot A_{j,i}^{1/p} \cdot c_i \\
&=: C_4 \cdot A_{j,i}^{1/p} \cdot c_i.
\end{aligned}$$

Here, Proposition 2.23 is applicable, since $\varphi_i \in C_c^\infty(\mathbb{R}^d)$ and $\widehat{\gamma^{(j)}} \in C^\infty(\mathbb{R}^d)$, so that $\varphi_i \cdot \widehat{\gamma^{(j)}} \in C_c^\infty(\mathbb{R}^d)$ and since clearly $\text{supp} [\varphi_i \widehat{\gamma^{(j)}}] \subset \overline{Q_i^*}$ and $\text{supp} [\varphi_i^* \widehat{f}] \subset \overline{Q_i^*}$.

Consequently, we arrive at

$$\sum_{i \in I} \left\| \mathcal{F}^{-1} \left(\widehat{\gamma^{(j)}} \cdot \varphi_i \cdot \widehat{f} \right) \right\|_{V_j}^p \leq C_4^p \cdot \sum_{i \in I} [A_{j,i} \cdot c_i^p] = C_4^p \cdot (\vec{A} \cdot e)_j < \infty, \quad (3.7)$$

since $\vec{A} : \ell_{w^{\min\{1,p\}}}^r(I) \rightarrow \ell_{w^{\min\{1,p\}}}^r(I)$ is well-defined and bounded and $e \in \ell_{w^{\min\{1,p\}}}^r(I)$.

Finally, we use the p -triangle inequality for $L^p(\mathbb{R}^d)$ (yielding the p -triangle inequality for $V_j = W_{T_j^{-T}[-1,1]^d}(L_v^p)$) to conclude that $F_j := \sum_{i \in I} \mathcal{F}^{-1} \left(\widehat{\gamma^{(j)}} \cdot \varphi_i \cdot \widehat{f} \right) \in V_j$ is well-defined, with normal convergence in V_j and a.e. absolute convergence of the defining series and with $\|F_j\|_{V_j} \leq C_4 \cdot (\vec{A} \cdot e)_j^{1/p}$.

Our next goal is to show that the previous results imply that $f_j \in \mathcal{S}'(\mathbb{R}^d)$ yields a well-defined tempered distribution. To this end, recall from Lemma 3.3 that $V_j \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$ for all $p \in (0, \infty]$. Consequently, we get $F_j \in \mathcal{S}'(\mathbb{R}^d)$ and $F_j = \sum_{i \in I} \mathcal{F}^{-1} \left(\widehat{\gamma^{(j)}} \cdot \varphi_i \cdot \widehat{f} \right)$ with unconditional convergence in $V_j \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$, which implies for $\phi \in \mathcal{S}(\mathbb{R}^d)$ that

$$\begin{aligned}
\langle \mathcal{F} F_j, \mathcal{F}^{-1} \phi \rangle_{\mathcal{S}', \mathcal{S}} &= \langle F_j, \phi \rangle_{\mathcal{S}', \mathcal{S}} = \sum_{i \in I} \left\langle \mathcal{F}^{-1} \left(\widehat{\gamma^{(j)}} \cdot \varphi_i \cdot \widehat{f} \right), \phi \right\rangle_{\mathcal{S}', \mathcal{S}}, \\
&= \sum_{i \in I} \left\langle \widehat{\gamma^{(j)}} \cdot \widehat{f}, \varphi_i \cdot \mathcal{F}^{-1} \phi \right\rangle_{\mathcal{D}'(\mathcal{O}), C_c^\infty(\mathcal{O})} \\
&= \langle f_j, \mathcal{F}^{-1} \phi \rangle_{\mathcal{S}', \mathcal{S}},
\end{aligned}$$

where the right-hand side is well-defined (with absolute convergence of the series), since the left-hand side is. This shows that $f_j = \mathcal{F} F_j \in \mathcal{F} V_j \subset \mathcal{S}'(\mathbb{R}^d)$ is a well-defined tempered distribution, as claimed. Finally, we have $\mathcal{F}^{-1} f_j = F_j = \sum_{i \in I} \mathcal{F}^{-1} \left(\widehat{\gamma^{(j)}} \cdot \varphi_i \cdot \widehat{f} \right)$, where the series converges normally in V_j and absolutely a.e., as claimed.

It remains to verify boundedness of Ana_Γ . But for $p \in [1, \infty]$, we have by solidity of $\ell_w^q(I)$ and by the triangle inequality for $L^p(\mathbb{R}^d)$, and since $C_1 = 1$ for $p \in [1, \infty]$, that

$$\begin{aligned}
\left\| \left(\left\| \gamma^{(j)} * f \right\|_{V_j} \right)_{j \in I} \right\|_{\ell_w^q} &= \left\| \left(\|F_j\|_{L_v^p} \right)_{j \in I} \right\|_{\ell_w^q} \leq \left\| \left(\sum_{i \in I} \left\| \mathcal{F}^{-1} \left(\widehat{\gamma^{(j)}} \cdot \varphi_i \cdot \widehat{f} \right) \right\|_{L_v^p} \right)_{j \in I} \right\|_{\ell_w^q} \\
&\stackrel{(\text{eq. (3.6)})}{\leq} \left\| \left[(\vec{A} \cdot c)_j \right]_{j \in I} \right\|_{\ell_w^q} \\
&\leq \|\vec{A}\| \cdot \|c\|_{\ell_w^q} \\
&\leq \|\Gamma_Q\| \cdot \|\vec{A}\| \cdot \|f\|_{\mathcal{D}(Q, L_v^p, \ell_w^q)} < \infty,
\end{aligned}$$

as desired.

Finally, in case of $p \in (0, 1)$, the p -triangle inequality for $W_{T_j^{-T}[-1,1]^d}(L_v^p)$ yields

$$\begin{aligned} \left\| \gamma^{(j)} * f \right\|_{V_j} &= \|F_j\|_{W_{T_j^{-T}[-1,1]^d}(L_v^p)} \leq \left[\sum_{i \in I} \left\| \mathcal{F}^{-1} \left(\widehat{\gamma^{(j)}} \cdot \varphi_i \cdot \widehat{f} \right) \right\|_{W_{T_j^{-T}[-1,1]^d}(L_v^p)}^p \right]^{1/p} \\ &\stackrel{(\text{eq. (3.7)})}{\leq} C_4 \cdot \left(\vec{A} \cdot e \right)_j^{1/p}. \end{aligned}$$

By solidity of $\ell_w^q(I)$, this implies

$$\begin{aligned} \left\| \left(\left\| \gamma^{(j)} * f \right\|_{V_j} \right)_{j \in I} \right\|_{\ell_w^q} &\leq C_4 \cdot \left\| \left(\vec{A} \cdot e \right)^{1/p} \right\|_{\ell_w^q} \\ &= C_4 \cdot \left\| \left(w^p \cdot [\vec{A} \cdot e] \right)^{1/p} \right\|_{\ell^q} \\ &= C_4 \cdot \left\| w^{\min\{1,p\}} \cdot [\vec{A} \cdot e] \right\|_{\ell^{q/p}}^{1/p} \\ &= C_4 \cdot \left\| \vec{A} \cdot e \right\|_{\ell_{w^{\min\{1,p\}}}^r}^{1/p} \\ &\leq C_4 \cdot \left\| \vec{A} \right\|^{1/p} \cdot \|e\|_{\ell_{w^{\min\{1,p\}}}^r}^{1/p} \\ &= C_4 \cdot \left\| \vec{A} \right\|^{1/p} \cdot \|c\|_{\ell_w^q} \\ &\leq C_1 C_4 \cdot \left\| \Gamma_Q \right\| \cdot \left\| \vec{A} \right\|^{1/p} \cdot \|f\|_{\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)} < \infty, \end{aligned}$$

which completes the proof. \square

Next, we establish the estimate “ \lesssim ” in equation (3.4), under suitable assumptions on $(\gamma_i)_{i \in I}$. Notice that up to now we have not excluded the case $\gamma_i \equiv 0$ for all $i \in I$. But if equation (3.4) was true, we would need at least that the family of frequency supports $\text{supp } \widehat{\gamma^{(i)}}$, with $i \in I$, covers all of \mathcal{O} . To ensure this, we introduce the following additional assumption:

Assumption 3.6. We assume that for each $i \in I$ there is some function $\theta_i^{\natural} \in C_c^\infty(\mathbb{R}^d)$ such that the family $\theta = (\theta_i^{\natural})_{i \in I}$ satisfies the following properties:

- (1) We have $\theta_i^{\natural} \cdot \widehat{\gamma_i} \equiv 1$ on Q'_i (and thus on $\overline{Q'_i}$) for all $i \in I$.
- (2) For each $p \in (0, \infty]$, the constant

$$\Omega_2^{(p,K)} := \Omega_2^{(p,K)}(\theta) := \begin{cases} \sup_{i \in I} \left\| \mathcal{F}^{-1} \theta_i^{\natural} \right\|_{W_{[-1,1]^d}(L_{(1+|\bullet|)^K}^p)}, & \text{if } p \in (0, 1), \\ \sup_{i \in I} \left\| \mathcal{F}^{-1} \theta_i^{\natural} \right\|_{L_{(1+|\bullet|)^K}^1}, & \text{if } p \in [1, \infty] \end{cases}$$

is finite.

We fix such a family $\theta = (\theta_i^{\natural})_{i \in I}$ and the constant $\Omega_2^{(p,K)}$ for the remainder of the paper. Finally, we recall $S_i \xi = T_i \xi + b_i$ and define

$$\theta_i := \theta_i^{\natural} \circ S_i^{-1} \in C_c^\infty(\mathbb{R}^d) \quad \forall i \in I. \quad \blacktriangleleft$$

At least in the case where the set of prototypes $\{\gamma_i \mid i \in I\}$ is finite, the preceding assumption can be heavily simplified, as we show now:

Lemma 3.7. Assume that there are N functions $\gamma_1^{(0)}, \dots, \gamma_N^{(0)}$ such that for each $i \in I$ we have $\gamma_i = \gamma_{n_i}^{(0)}$ for a suitable $n_i \in \underline{N}$. For $n \in \underline{N}$ let

$$Q^{(n)} := \bigcup \{Q'_i \mid i \in I \text{ and } n_i = n\}.$$

If there is some $c > 0$ satisfying $|(\mathcal{F}\gamma_n^{(0)})(\xi)| \geq c$ for all $\xi \in Q^{(n)}$, then the family $(\gamma_i)_{i \in I}$ satisfies Assumption 3.6.

In fact, for arbitrary $p_0 \in (0, 1]$ and $K^{(0)} \geq 0$, there is a constant $\Omega_3 = \Omega_3(\mathcal{Q}, \gamma_1^{(0)}, \dots, \gamma_N^{(0)}, p_0, K^{(0)}, d) > 0$ satisfying

$$\Omega_2^{(p,K)} \leq \Omega_3 \quad \forall p \geq p_0 \text{ and } K \leq K^{(0)}. \quad \blacktriangleleft$$

Remark. If $\gamma_i = \gamma$ for all $i \in I$, then the above assumptions reduce to $|\widehat{\gamma}(\xi)| \geq c > 0$ for all $\xi \in Q := \bigcup_{i \in I} Q'_i$. \blacklozenge

Proof. Recall from Assumption 3.1 that we always have $\widehat{\gamma}_i \in C^\infty(\mathbb{R}^d)$. Now, by possibly dropping some elements of the family $\gamma_1^{(0)}, \dots, \gamma_N^{(0)}$, we can assume that for each $n \in \underline{N}$, there is some $i \in I$ satisfying $n_i = n$ and thus $\gamma_n^{(0)} = \gamma_i$. In particular, this implies $\widehat{\gamma_n^{(0)}} \in C^\infty(\mathbb{R}^d)$ for all $n \in \underline{N}$.

By continuity of $\mathcal{F}\gamma_n^{(0)}$, we get $|(\mathcal{F}\gamma_n^{(0)})(\xi)| \geq c$ for all $\xi \in \overline{Q^{(n)}}$. Furthermore, recall from Subsection 1.3 that we have $Q'_i \subset \overline{B_{R_Q}}(0)$ for all $i \in I$, so that each of the sets $Q^{(n)}$ is bounded. Hence, $\overline{Q^{(n)}}$ is compact. Again by continuity of $\mathcal{F}\gamma_n^{(0)}$, each of the sets

$$U_n := \left\{ \xi \in \mathbb{R}^d \mid |(\mathcal{F}\gamma_n^{(0)})(\xi)| > \frac{c}{2} \right\}$$

is open with $\overline{Q^{(n)}} \subset U_n$. Thus, the C^∞ -Urysohn-Lemma (cf. [29, Lemma 8.18]) yields some $\eta_n \in C_c^\infty(U_n)$ with $\eta_n|_{Q^{(n)}} \equiv 1$.

Now, note that $\eta_n / \widehat{\gamma_n^{(0)}} \in C_c^\infty(U_n)$ is well-defined, since $\widehat{\gamma_n^{(0)}} \neq 0$ on U_n . Thus, the function

$$\theta^{(n)} : \mathbb{R}^d \rightarrow \mathbb{C}, \xi \mapsto \begin{cases} \frac{\eta_n(\xi)}{\widehat{\gamma_n^{(0)}}(\xi)}, & \text{if } \xi \in U_n, \\ 0, & \text{if } \xi \notin U_n \end{cases}$$

is a smooth function $\theta^{(n)} \in C_c^\infty(\mathbb{R}^d)$ with $\text{supp } \theta^{(n)} \subset U_n$ and with $\widehat{\theta^{(n)}} \cdot \widehat{\gamma_n^{(0)}} = \eta_n \equiv 1$ on $Q^{(n)}$.

Now, define $\theta_i^\sharp := \theta^{(n_i)} \in C_c^\infty(\mathbb{R}^d)$ for $i \in I$. Then, for each $i \in I$, we have $\theta_i^\sharp \cdot \widehat{\gamma_i} = \widehat{\theta^{(n_i)}} \cdot \widehat{\gamma_{n_i}^{(0)}} \equiv 1$ on $Q^{(n_i)} \supset Q'_i$, cf. the definition of $Q^{(n)}$.

Finally, Lemma 2.3 (with $N = K^{(0)} + \frac{d}{p_0} + 1$) yields for $p \geq p_0$ and $K \leq K^{(0)}$ the estimate

$$\begin{aligned} \left\| \mathcal{F}^{-1} \theta_i^\sharp \right\|_{L^p_{(1+|\cdot|)^K}} &\leq \left\| \mathcal{F}^{-1} \theta_i^\sharp \right\|_{W_{[-1,1]^d}(L^p_{(1+|\cdot|)^K})} \leq \left(1 + 2\sqrt{d}\right)^N \cdot \left(\frac{1}{p} \frac{s_d}{N - K - \frac{d}{p}}\right)^{1/p} \cdot \left\| \mathcal{F}^{-1} \theta_i^\sharp \right\|_N \\ &\leq \left(1 + 2\sqrt{d}\right)^N \cdot \left(1 + \frac{s_d}{p_0}\right)^{1/p_0} \cdot \max_{n \in \underline{N}} \left\| \mathcal{F}^{-1} \theta^{(n)} \right\|_N =: \Omega_3. \end{aligned}$$

Since N only depends on $K^{(0)}, d, p_0$ and since $\theta^{(1)}, \dots, \theta^{(N)}$ only depend on \mathcal{Q} and on $\gamma_1^{(0)}, \dots, \gamma_N^{(0)}$, Ω_3 is as claimed in the lemma. Note that each of the norms $\left\| \mathcal{F}^{-1} \theta^{(n)} \right\|_N$ is finite, since $\theta^{(n)} \in C_c^\infty(\mathbb{R}^d)$, from which we get $\mathcal{F}^{-1} \theta^{(n)} \in \mathcal{S}(\mathbb{R}^d)$. \square

Now, instead of just establishing equation (3.4), we will actually show that the “coefficient map” Ana_Γ from Theorem 3.5 yields a semi-discrete Banach frame for $\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$. By this we mean that there exists a bounded linear “reconstruction” map $R : V \rightarrow \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$ satisfying $R \circ \text{Ana}_\Gamma = \text{id}_{\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)}$. For the construction of R , the following result will turn out to be helpful:

Lemma 3.8. *Assume that $\Gamma = (\gamma_i)_{i \in I}$ satisfies Assumption 3.6 and let $(\theta_i)_{i \in I}$ be defined as in that assumption.*

Then $\widehat{\gamma^{(i)}} \cdot \theta_i \equiv 1$ on $\overline{Q_i}$ for each $i \in I$ and each of the maps

$$I_i : V_i \rightarrow V_i, f \mapsto (\mathcal{F}^{-1} \theta_i) * f$$

is well-defined and bounded, with $\sup_{i \in I} \|I_i\| \leq C < \infty$, where

$$C := \begin{cases} \Omega_0^{4K} \Omega_1^4 \Omega_2^{(p,K)} \cdot d^{-\frac{d}{2p}} \cdot (972 \cdot d^{5/2})^{K + \frac{d}{p}}, & \text{if } p \in (0, 1), \\ \Omega_0^K \Omega_1 \Omega_2^{(p,K)}, & \text{if } p \in [1, \infty]. \end{cases}$$

Hence, the map

$$m_\theta := \bigotimes_{i \in I} I_i : V \rightarrow V, (f_i)_{i \in I} \mapsto ((\mathcal{F}^{-1} \theta_i) * f_i)_{i \in I}$$

is well-defined and bounded as well, with $\|m_\theta\| \leq C$. \blacktriangleleft

Proof. First, observe

$$\widehat{\gamma^{(i)}} \cdot \theta_i = (\widehat{\gamma_i} \circ S_i^{-1}) \cdot (\theta_i^\sharp \circ S_i^{-1}) = \underbrace{(\widehat{\gamma_i} \cdot \theta_i^\sharp)}_{\equiv 1 \text{ on } \overline{Q'_i}} \circ S_i^{-1} \equiv 1 \text{ on } S_i \overline{Q'_i} = \overline{Q_i},$$

so that it remains to show that each of the maps I_i is well-defined and bounded, with the claimed estimate for the operator norm.

In case of $p \in [1, \infty]$, this is a consequence of equation (1.12), once we show that $\|\mathcal{F}^{-1}\theta_i\|_{L^1_{v_0}}$ is uniformly bounded. But we simply have

$$\theta_i = \theta_i^{\natural} \circ S_i^{-1} = L_{b_i} \left(\theta_i^{\natural} \circ T_i^{-1} \right) \quad \text{and hence} \quad \mathcal{F}^{-1}\theta_i = |\det T_i| \cdot M_{b_i} \left[\left(\mathcal{F}^{-1}\theta_i^{\natural} \right) \circ T_i^T \right], \quad (3.8)$$

which implies

$$\begin{aligned} \|\mathcal{F}^{-1}\theta_i\|_{L^1_{v_0}} &= |\det T_i| \cdot \|v_0 \cdot \left[\left(\mathcal{F}^{-1}\theta_i^{\natural} \right) \circ T_i^T \right]\|_{L^1} \\ &= \|(v_0 \circ T_i^{-T}) \cdot \left(\mathcal{F}^{-1}\theta_i^{\natural} \right)\|_{L^1} \\ (\text{assumption on } v_0) &\leq \Omega_1 \cdot \|x \mapsto (1 + |T_i^{-T}x|)^K \cdot \left(\mathcal{F}^{-1}\theta_i^{\natural} \right)(x)\|_{L^1} \\ (\text{eq. (1.11)}) &\leq \Omega_0^K \Omega_1 \cdot \|(1 + |\bullet|)^K \cdot \mathcal{F}^{-1}\theta_i^{\natural}\|_{L^1} \leq \Omega_0^K \Omega_1 \cdot \Omega_2^{(p,K)}. \end{aligned}$$

Finally, for $p \in (0, 1)$, we get from Corollary 2.22 for $C_1 := \Omega_0^{3K} \Omega_1^3 \cdot d^{-\frac{d}{2p}} \cdot (972 \cdot d^{5/2})^{K+\frac{d}{p}}$ that

$$\begin{aligned} \|(\mathcal{F}^{-1}\theta_i) * f\|_{W_{T_i^{-T}[-1,1]^d}(L^p_v)} &\leq C_1 \cdot |\det T_i|^{\frac{1}{p}-1} \cdot \|\mathcal{F}^{-1}\theta_i\|_{W_{T_i^{-T}[-1,1]^d}(L^p_{v_0})} \cdot \|f\|_{W_{T_i^{-T}[-1,1]^d}(L^p_v)} \\ (\text{eq. (3.8) and } \|M_b f\|_{W_Q(L^p_{v_0})} = \|f\|_{W_Q(L^p_{v_0})}) &= C_1 \cdot |\det T_i|^{\frac{1}{p}-1} \cdot |\det T_i| \cdot \left\| \left(\mathcal{F}^{-1}\theta_i^{\natural} \right) \circ T_i^T \right\|_{W_{T_i^{-T}[-1,1]^d}(L^p_{v_0})} \cdot \|f\|_{V_i} \\ (\text{Lemma 2.4}) &= C_1 \cdot |\det T_i|^{\frac{1}{p}} \cdot \left\| \left(M_{[-1,1]^d} \left[\mathcal{F}^{-1}\theta_i^{\natural} \right] \right) \circ T_i^T \right\|_{L^p_{v_0}} \cdot \|f\|_{V_i} \\ &= C_1 \cdot \left\| (v_0 \circ T_i^{-T}) \cdot M_{[-1,1]^d} \left[\mathcal{F}^{-1}\theta_i^{\natural} \right] \right\|_{L^p} \cdot \|f\|_{V_i} \\ (\text{assumption on } v_0 \text{ and eq. (1.11)}) &\leq \Omega_0^K \Omega_1 C_1 \cdot \|(1 + |\bullet|)^K \cdot M_{[-1,1]^d} \left[\mathcal{F}^{-1}\theta_i^{\natural} \right]\|_{L^p} \cdot \|f\|_{V_i} \\ &\leq \Omega_0^K \Omega_1 \Omega_2^{(p,K)} \cdot C_1 \cdot \|f\|_{V_i}. \quad \square \end{aligned}$$

Our final ingredient for the construction of the “reconstruction map” $R : V \rightarrow \mathcal{D}(\mathcal{Q}, L^p_v, \ell^q_w)$ is the following lemma.

Lemma 3.9. *The map*

$$\text{Synth}_{\mathcal{D}} : V \rightarrow \mathcal{D}(\mathcal{Q}, L^p_v, \ell^q_w), (f_i)_{i \in I} \mapsto \sum_{i \in I} [(\mathcal{F}^{-1}\varphi_i) * f_i] \stackrel{\text{Lemma 3.3}}{=} \sum_{i \in I} [\mathcal{F}^{-1}(\varphi_i \cdot \widehat{f_i})]$$

is well-defined and bounded with unconditional convergence of the series in $Z'(\mathcal{O})$ and with $\|\text{Synth}_{\mathcal{D}}\| \leq \|\Gamma_{\mathcal{Q}}\| \cdot C$, where

$$C = \begin{cases} \frac{(768/\sqrt{d})^{d/p}}{59049 \cdot 12^d \cdot d^5} \cdot \left(186624 \cdot d^4 \cdot \left\lceil K + \frac{d+1}{p} \right\rceil \right)^{1+\lceil K + \frac{d+1}{p} \rceil} \cdot (1 + R_{\mathcal{Q}} C_{\mathcal{Q}})^{d(\frac{2}{p}-1)} \cdot \Omega_0^{4K} \Omega_1^4 \cdot N_{\mathcal{Q}}^{\frac{1}{p}-1} C_{\mathcal{Q}, \Phi, v_0, p}^2, & \text{if } p \in (0, 1), \\ C_{\mathcal{Q}, \Phi, v_0, p}^2, & \text{if } p \in [1, \infty] \end{cases}$$

and where $\Gamma_{\mathcal{Q}} : \ell^q_w(I) \rightarrow \ell^q_w(I)$, $c \mapsto c^*$ denotes the \mathcal{Q} -clustering map, i.e., $c_i^* = \sum_{\ell \in i^*} c_{\ell}$. \blacktriangleleft

Proof. Recall that the Fourier transform $\mathcal{F} : Z'(\mathcal{O}) \rightarrow \mathcal{D}'(\mathcal{O})$ is an isomorphism that restricts to an isometric isomorphism $\mathcal{F} : \mathcal{D}(\mathcal{Q}, L^p_v, \ell^q_w) \rightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p_v, \ell^q_w)$. Hence, it suffices to show that the map

$$\Theta := \mathcal{F} \circ \text{Synth}_{\mathcal{D}} : V \rightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p_v, \ell^q_w), (f_i)_{i \in I} \mapsto \sum_{i \in I} \varphi_i \widehat{f_i}$$

is well-defined and bounded, with unconditional convergence of the series in $\mathcal{D}'(\mathcal{O})$. Since the $(\varphi_i)_{i \in I}$ form a *locally finite* partition of unity on \mathcal{O} , the series *does* converge unconditionally in $\mathcal{D}'(\mathcal{O})$, given that each term is a well-defined element of $\mathcal{D}'(\mathcal{O})$. But this is an easy consequence of the embedding $V_i \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$, which holds for each $i \in I$, according to Lemma 3.3.

Now, define

$$c_i := \|f_i\|_{V_i} = \begin{cases} \|f_i\|_{L^p_v}, & \text{if } p \in [1, \infty], \\ \|f_i\|_{W_{T_i^{-T}[-1,1]^d}(L^p_v)}, & \text{if } p \in (0, 1). \end{cases}$$

By definition of V , we have $c := (c_i)_{i \in I} \in \ell_w^q(I)$ and $\|(f_i)_{i \in I}\|_V = \|c\|_{\ell_w^q}$. Furthermore, since the \mathcal{Q} -clustering map $\Gamma_{\mathcal{Q}}$ is bounded, it suffices to show $\|\mathcal{F}^{-1}(\varphi_j \cdot \Theta(f_i)_{i \in I})\|_{L_v^p} \leq C_1 \cdot c_j^*$ for all $j \in I$ and a suitable constant $C_1 > 0$, since this implies

$$\|\Theta(f_i)_{i \in I}\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L_v^p, \ell_w^q)} = \left\| \left(\|\mathcal{F}^{-1}(\varphi_j \cdot \Theta(f_i)_{i \in I})\|_{L_v^p} \right)_{j \in I} \right\|_{\ell_w^q} \leq C_1 \cdot \|c^*\|_{\ell_w^q} \leq C_1 \|\Gamma_{\mathcal{Q}}\| \cdot \|c\|_{\ell_w^q} = C_1 \|\Gamma_{\mathcal{Q}}\| \cdot \|(f_i)_{i \in I}\|_V.$$

To show $\|\mathcal{F}^{-1}(\varphi_j \cdot \Gamma(f_i)_{i \in I})\|_{L_v^p} \leq C_1 \cdot c_j^*$, we distinguish two cases regarding p :

Let us start with the case $p \in [1, \infty]$. Since $\varphi_j \varphi_\ell \equiv 0$ unless $\ell \in j^*$, we have

$$\begin{aligned} \|\mathcal{F}^{-1}[\varphi_j \cdot \Theta(f_\ell)_{\ell \in I}]\|_{L_v^p} &= \left\| \mathcal{F}^{-1} \left[\varphi_j \cdot \sum_{\ell \in I} \varphi_\ell \widehat{f}_\ell \right] \right\|_{L_v^p} \\ &= \left\| \sum_{\ell \in j^*} \mathcal{F}^{-1}[\varphi_j \varphi_\ell \widehat{f}_\ell] \right\|_{L_v^p} \\ &\leq \sum_{\ell \in j^*} \|(\mathcal{F}^{-1} \varphi_j) * (\mathcal{F}^{-1} \varphi_\ell) * f_\ell\|_{L_v^p} \\ &\stackrel{(\text{eq. (1.12)})}{\leq} \sum_{\ell \in j^*} \|\mathcal{F}^{-1} \varphi_j\|_{L_{v_0}^1} \|\mathcal{F}^{-1} \varphi_\ell\|_{L_{v_0}^1} \cdot \|f_\ell\|_{L_v^p} \\ &\leq C_{\mathcal{Q}, \Phi, v_0, p}^2 \cdot \sum_{\ell \in j^*} \|f_\ell\|_{L_v^p} = C_{\mathcal{Q}, \Phi, v_0, p}^2 \cdot c_j^*, \end{aligned}$$

so that we can choose $C_1 := C_{\mathcal{Q}, \Phi, v_0, p}^2$.

Now, we consider the case $p \in (0, 1)$. Here, we can first argue as before:

$$\begin{aligned} \|\mathcal{F}^{-1}[\varphi_j \cdot \Theta(f_\ell)_{\ell \in I}]\|_{L_v^p} &= \left\| \sum_{\ell \in j^*} \mathcal{F}^{-1}[\varphi_j \varphi_\ell \widehat{f}_\ell] \right\|_{L_v^p} \\ &\stackrel{(\text{quasi-triangle inequality and } |j^*| \leq N_{\mathcal{Q}})}{\leq} N_{\mathcal{Q}}^{\frac{1}{p}-1} \cdot \sum_{\ell \in j^*} \|\mathcal{F}^{-1}(\varphi_j \varphi_\ell) * f_\ell\|_{L_v^p} \\ &\stackrel{(\text{Lemma 2.2})}{\leq} N_{\mathcal{Q}}^{\frac{1}{p}-1} \cdot \sum_{\ell \in j^*} \|\mathcal{F}^{-1}(\varphi_j \varphi_\ell) * f_\ell\|_{W_{T_\ell^{-T}[-1,1]^d}(L_v^p)}. \end{aligned}$$

Then, we set $C_2 := \Omega_0^{3K} \Omega_1^3 \cdot d^{-\frac{d}{2p}} \cdot (972 \cdot d^{5/2})^{K+\frac{d}{p}}$ and use Corollary 2.22 to estimate each summand as follows:

$$\begin{aligned} &\|\mathcal{F}^{-1}(\varphi_j \varphi_\ell) * f_\ell\|_{W_{T_\ell^{-T}[-1,1]^d}(L_v^p)} \\ &\stackrel{(\text{Corollary 2.22})}{\leq} C_2 \cdot |\det T_\ell|^{\frac{1}{p}-1} \cdot \|\mathcal{F}^{-1}(\varphi_j \varphi_\ell)\|_{W_{T_\ell^{-T}[-1,1]^d}(L_{v_0}^p)} \|f_\ell\|_{W_{T_\ell^{-T}[-1,1]^d}(L_v^p)} \\ &= C_2 \cdot |\det T_\ell|^{\frac{1}{p}-1} \cdot \|\mathcal{F}^{-1}(\varphi_j \varphi_\ell)\|_{W_{T_\ell^{-T}[-1,1]^d}(L_{v_0}^p)} \cdot c_\ell. \end{aligned}$$

Now, note

$$\text{supp}(\varphi_j \varphi_\ell) \subset \overline{Q_\ell} = T_\ell \overline{Q'_\ell} + b_\ell \subset T_\ell [-R_{\mathcal{Q}}, R_{\mathcal{Q}}]^d + b_\ell,$$

so that Theorem 2.17 (with v_0 instead of v) shows for

$$C_3 := 2^{4(1+\frac{d}{p})} s_d^{\frac{1}{p}} \left(192 \cdot d^{3/2} \cdot \left\lceil K + \frac{d+1}{p} \right\rceil \right)^{\left\lceil K + \frac{d+1}{p} \right\rceil + 1} \cdot \Omega_0^K \Omega_1 \cdot (1 + R_{\mathcal{Q}})^{\frac{d}{p}}$$

that

$$\|\mathcal{F}^{-1}(\varphi_j \varphi_\ell)\|_{W_{T_\ell^{-T}[-1,1]^d}(L_{v_0}^p)} \leq C_3 \cdot \|\mathcal{F}^{-1}(\varphi_j \varphi_\ell)\|_{L_{v_0}^p}$$

$$\begin{aligned} &(\text{Proposition 2.23 and } \text{supp } \varphi_j \subset \overline{Q_j} \subset \overline{Q_j^*} \text{ and } \text{supp } \varphi_\ell \subset \overline{Q_\ell} \subset \overline{Q_\ell^*}) \leq C_3 (12R_{\mathcal{Q}}C_{\mathcal{Q}})^{d(\frac{1}{p}-1)} \cdot |\det T_j|^{\frac{1}{p}-1} \cdot \|\mathcal{F}^{-1} \varphi_j\|_{L_{v_0}^p} \cdot \|\mathcal{F}^{-1} \varphi_\ell\|_{L_{v_0}^p} \\ &\leq C_4 \cdot \|\mathcal{F}^{-1} \varphi_\ell\|_{L_{v_0}^p} \end{aligned}$$

for $C_4 := C_3 \cdot (12R_{\mathcal{Q}}C_{\mathcal{Q}})^{d(\frac{1}{p}-1)} \cdot C_{\mathcal{Q},\Phi,v_0,p}$. In total, we conclude

$$\begin{aligned} \|\mathcal{F}^{-1}(\varphi_j \varphi_\ell) * f_\ell\|_{W_{T_\ell^{-T}[-1,1]^d}(L_v^p)} &\leq C_2 \cdot |\det T_\ell|^{\frac{1}{p}-1} \cdot \|\mathcal{F}^{-1}(\varphi_j \varphi_\ell)\|_{W_{T_\ell^{-T}[-1,1]^d}(L_{v_0}^p)} \cdot c_\ell \\ &\leq C_2 C_4 \cdot |\det T_\ell|^{\frac{1}{p}-1} \cdot \|\mathcal{F}^{-1} \varphi_\ell\|_{L_{v_0}^p} \cdot c_\ell \\ &\leq C_2 C_4 C_{\mathcal{Q},\Phi,v_0,p} \cdot c_\ell \end{aligned}$$

and hence

$$\begin{aligned} \|\mathcal{F}^{-1}[\varphi_j \cdot \Theta(f_\ell)_{\ell \in I}]\|_{L_v^p} &\leq N_{\mathcal{Q}}^{\frac{1}{p}-1} \cdot \sum_{\ell \in j^*} \|\mathcal{F}^{-1}(\varphi_j \varphi_\ell) * f_\ell\|_{W_{T_\ell^{-T}[-1,1]^d}(L_v^p)} \\ &\leq N_{\mathcal{Q}}^{\frac{1}{p}-1} C_2 C_4 C_{\mathcal{Q},\Phi,v_0,p} \cdot \sum_{\ell \in j^*} c_\ell \\ &= N_{\mathcal{Q}}^{\frac{1}{p}-1} C_2 C_4 C_{\mathcal{Q},\Phi,v_0,p} \cdot c_j^*, \end{aligned}$$

so that the desired estimate from the start of the proof holds with $C_1 := N_{\mathcal{Q}}^{\frac{1}{p}-1} C_2 C_4 C_{\mathcal{Q},\Phi,v_0,p}$. Now, we finally set $N := \left\lceil K + \frac{d+1}{p} \right\rceil$ and observe because of $N \geq K + \frac{d}{p} + 1 \geq \frac{d}{p} + 1$ and $s_d \leq 4^d$, as well as $C_{\mathcal{Q}} \geq \|T_i^{-1} T_i\| \geq 1$ that

$$\begin{aligned} C_1 &= d^{-\frac{d}{2p}} \cdot \left(972 \cdot d^{\frac{5}{2}}\right)^{K+\frac{d}{p}} 2^{4(1+\frac{d}{p})} s_d^{\frac{1}{p}} \left(192 \cdot d^{\frac{3}{2}} \cdot N\right)^{N+1} \cdot (1+R_{\mathcal{Q}})^{\frac{d}{p}} (12R_{\mathcal{Q}}C_{\mathcal{Q}})^{d(\frac{1}{p}-1)} \cdot \Omega_0^{4K} \Omega_1^4 \cdot N_{\mathcal{Q}}^{\frac{1}{p}-1} C_{\mathcal{Q},\Phi,v_0,p}^2 \\ &\leq 2^4 \cdot \left(2^6/\sqrt{d}\right)^{d/p} \left(972 \cdot d^{\frac{5}{2}}\right)^{K+\frac{d}{p}} \left(192 \cdot d^{\frac{3}{2}} \cdot N\right)^{N+1} \cdot 12^{d(\frac{1}{p}-1)} (1+R_{\mathcal{Q}}C_{\mathcal{Q}})^{d(\frac{2}{p}-1)} \cdot \Omega_0^{4K} \Omega_1^4 \cdot N_{\mathcal{Q}}^{\frac{1}{p}-1} C_{\mathcal{Q},\Phi,v_0,p}^2 \\ &\leq \frac{(768/\sqrt{d})^{d/p}}{59049 \cdot 12^d \cdot d^5} \cdot (186624 \cdot d^4 \cdot N)^{N+1} \cdot (1+R_{\mathcal{Q}}C_{\mathcal{Q}})^{d(\frac{2}{p}-1)} \cdot \Omega_0^{4K} \Omega_1^4 \cdot N_{\mathcal{Q}}^{\frac{1}{p}-1} C_{\mathcal{Q},\Phi,v_0,p}^2. \quad \square \end{aligned}$$

Now, we can finally show existence of the reconstruction map R and thus also derive the estimate “ \lesssim ” in equation (3.4).

Theorem 3.10. Assume that the family $\Gamma = (\gamma_i)_{i \in I}$ satisfies Assumptions 3.1 and 3.6.

Then, with m_θ as in Lemma 3.8, with $\text{Synth}_{\mathcal{D}}$ as in Lemma 3.9 and with Ana_Γ as in Theorem 3.5, the map

$$R := \text{Synth}_{\mathcal{D}} \circ m_\theta : V \rightarrow \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$$

is well-defined and bounded with $\|R\| \leq \|\Gamma_{\mathcal{Q}}\| C_{\mathcal{Q},\Phi,v_0,p}^2 \cdot C$ for

$$C := \begin{cases} \frac{(768/d)^{d/p}}{2^{35} \cdot 12^d \cdot d^{10}} \cdot \left(2^{28} \cdot d^{\frac{13}{2}} \cdot \left\lceil K + \frac{d+1}{p} \right\rceil\right)^{1+\left\lceil K + \frac{d+1}{p} \right\rceil} \cdot (1+R_{\mathcal{Q}}C_{\mathcal{Q}})^{d(\frac{2}{p}-1)} \cdot \Omega_0^{8K} \Omega_1^8 \Omega_2^{(p,K)} \cdot N_{\mathcal{Q}}^{\frac{1}{p}-1}, & \text{if } p < 1, \\ \Omega_0^K \Omega_1 \Omega_2^{(p,K)}, & \text{if } p \geq 1, \end{cases}$$

where as usual $\Gamma_{\mathcal{Q}} : \ell_w^q(I) \rightarrow \ell_w^q(I)$ denotes the \mathcal{Q} -clustering map.

Furthermore, R satisfies

$$R \circ \text{Ana}_\Gamma = \text{id}_{\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)}. \quad (3.9)$$

In particular, equation (3.4) is satisfied, i.e., we have

$$\|f\|_{\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)} \asymp \|(\gamma^{(i)} * f)_{i \in I}\|_V \quad \forall f \in \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q). \quad \blacktriangleleft$$

Proof. Boundedness of $\text{Synth}_{\mathcal{D}}$ and m_θ and thus of R is a consequence of Lemmas 3.9 and 3.8, respectively, so that it suffices to prove equation (3.9).

To see this, we again use the isomorphism $\mathcal{F} : \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q) \rightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L_v^p, \ell_w^q)$. Recall from Lemma 3.9 that we have

$$(\mathcal{F} \circ \text{Synth}_{\mathcal{D}})(f_i)_{i \in I} = \sum_{i \in I} \left(\varphi_i \cdot \widehat{f}_i \right) \quad \text{for} \quad (f_i)_{i \in I} \in V = \ell_w^q([V_i]_{i \in I}),$$

where it is used that $f_i \in V_i \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$ for all $i \in I$.

Hence, for $f \in \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$, we have

$$\begin{aligned} (\mathcal{F} \circ R \circ \text{Ana}_\Gamma) f &= \sum_{i \in I} [\varphi_i \cdot \mathcal{F}[(m_\theta \circ \text{Ana}_\Gamma) f]_i] \\ &= \sum_{i \in I} [\varphi_i \cdot \mathcal{F}[(\mathcal{F}^{-1} \theta_i) * (\gamma^{(i)} * f)]] \\ &\stackrel{(\text{Lemma 3.3})}{=} \sum_{i \in I} [\varphi_i \cdot \theta_i \cdot \widehat{\gamma^{(i)}} * f] \\ &\stackrel{(\text{Special Def. of } \gamma^{(i)} * f = \mathcal{F}^{-1} f_i, \text{ cf. Theorem 3.5})}{=} \sum_{i \in I} [\varphi_i \cdot \theta_i \cdot f_i], \end{aligned}$$

where

$$f_i : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathbb{C}, \phi \mapsto \sum_{\ell \in I} \langle \widehat{\gamma^{(i)}} \cdot \widehat{f}, \varphi_\ell \phi \rangle_{\mathcal{D}'(\mathcal{O}), C_c^\infty(\mathcal{O})}.$$

Thus, we have for arbitrary $\phi \in C_c^\infty(\mathcal{O})$ that

$$\begin{aligned} \langle (\mathcal{F} \circ R \circ \text{Ana}_\Gamma) f, \phi \rangle_{\mathcal{D}'(\mathcal{O}), C_c^\infty(\mathcal{O})} &= \sum_{i \in I} \langle f_i, \varphi_i \cdot \theta_i \cdot \phi \rangle_{\mathcal{S}', \mathcal{S}} \\ &= \sum_{i \in I} \sum_{\ell \in I} \langle \widehat{\gamma^{(i)}} \cdot \widehat{f}, \varphi_\ell \varphi_i \cdot \theta_i \cdot \phi \rangle_{\mathcal{D}'(\mathcal{O}), C_c^\infty(\mathcal{O})} \\ &= \sum_{i \in I} \sum_{\ell \in I} \langle \widehat{f}, \varphi_\ell \varphi_i \cdot \widehat{\gamma^{(i)}} \cdot \theta_i \cdot \phi \rangle_{\mathcal{D}'(\mathcal{O}), C_c^\infty(\mathcal{O})} \\ &\stackrel{(\widehat{\gamma^{(i)}} \cdot \theta_i \equiv 1 \text{ on } \overline{Q_i} \supset \text{supp } \varphi_i, \text{ cf. Lemma 3.8})}{=} \sum_{i \in I} \sum_{\ell \in I} \langle \widehat{f}, \varphi_\ell \varphi_i \cdot \phi \rangle_{\mathcal{D}'(\mathcal{O}), C_c^\infty(\mathcal{O})} \\ &\stackrel{(\phi \in C_c^\infty(\mathcal{O}) \text{ and } (\varphi_i)_{i \in I} \text{ loc. finite part. of unity on } \mathcal{O})}{=} \sum_{i \in I} \langle \widehat{f}, \varphi_i \cdot \phi \rangle_{\mathcal{D}'(\mathcal{O}), C_c^\infty(\mathcal{O})} \\ &\stackrel{(\text{as above})}{=} \langle \widehat{f}, \phi \rangle_{\mathcal{D}'(\mathcal{O}), C_c^\infty(\mathcal{O})}. \end{aligned}$$

Hence, we have shown $\mathcal{F} \circ R \circ \text{Ana}_\Gamma = \mathcal{F}$ on $\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$. Since $\mathcal{F} : \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q) \rightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L_v^p, \ell_w^q)$ is an isomorphism, this implies $R \circ \text{Ana}_\Gamma = \text{id}_{\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)}$, as desired. \square

4. FULLY DISCRETE BANACH FRAMES

In the previous section, we obtained *semi-discrete* Banach frames for $\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$; in particular, we showed

$$\|f\|_{\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)} \asymp \left\| \left(\|\gamma^{(i)} * f\|_{V_i} \right)_{i \in I} \right\|_{\ell_w^q}.$$

We call such a frame **semi-discrete**, because while the index set I is discrete, the convolutions $\gamma^{(i)} * f$ are treated as genuine functions, which are defined on the continuous (non-discrete) index set \mathbb{R}^d .

In this section, our aim is a further discretization of these frames, so that we will in the end obtain a (quasi)-norm equivalence of the form

$$\|f\|_{\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)} \asymp \left\| \left(\left\| \left(\gamma^{[j]} * f \right) (\delta \cdot T_j^{-T} k) \right\|_{C_j^{(\delta)}} \right)_{j \in I} \right\|_{\ell_{u_q}^q} \quad \text{for a suitable weight } u_q \text{ on } I, \quad (4.1)$$

where for each $\delta \in (0, 1]$ and $j \in I$, the **coefficient space** $C_j^{(\delta)}$ is given by

$$C_j^{(\delta)} := \ell_{v^{(j, \delta)}}^p(\mathbb{Z}^d) \quad \text{with} \quad v_k^{(j, \delta)} = v(\delta \cdot T_j^{-T} k) \quad \text{for } k \in \mathbb{Z}^d. \quad (4.2)$$

For simplicity, the reader should keep in mind the important special case $v \equiv 1$, for which $C_j^{(\delta)} = \ell^p(\mathbb{Z}^d)$, independently of j, δ .

To ensure that equation (4.1) holds, we will introduce suitable assumptions on $(\gamma_i)_{i \in I}$ and $\delta > 0$. In particular, for the formula above to make sense, we also have to establish (at least) continuity of $\gamma^{[j]} * f$ (and thus of $\gamma^{[j]} * f$), so that the pointwise evaluations $(\gamma^{[j]} * f)(\delta \cdot T_j^{-T} k)$ are meaningful; note that up to now, we only know (for $p \in [1, \infty]$) that $\gamma^{[j]} * f \in L_v^p(\mathbb{R}^d)$.

To ensure this continuity, we introduce a new set of additional assumptions and notations. In these assumptions, the $L_{v_0}^p$ (quasi)-norm of certain vector valued functions $g : \mathbb{R}^d \rightarrow \mathbb{C}^k$ is considered. This has to be understood as $\|g\|_{L_{v_0}^p} := \|\widehat{|g|}\|_{L_{v_0}^p}$, where as usual $|g|(x) := |g(x)| = \|g(x)\|_2$ denotes the euclidean norm of $g(x)$. Furthermore, for such a function $g = (g_1, \dots, g_k)$, expressions as the (inverse) Fourier transform $\mathcal{F}^{-1}g := (\mathcal{F}^{-1}g_1, \dots, \mathcal{F}^{-1}g_k)$ are always understood in a coordinatewise sense.

Assumption 4.1. In addition to Assumption 3.1, we assume the following:

- (1) We have $\gamma_i \in C^1(\mathbb{R}^d)$ for all $i \in I$ and the gradient $\phi_i := \nabla \gamma_i$ satisfies the following:
 - (a) ϕ_i is bounded for each $i \in I$,
 - (b) We have $\phi_i \in L_{v_0}^1(\mathbb{R}^d; \mathbb{C}^d) \hookrightarrow L^1(\mathbb{R}^d; \mathbb{C}^d)$ for all $i \in I$,
 - (c) We have $\widehat{\phi_i} \in C^\infty(\mathbb{R}^d; \mathbb{C}^d)$ for all $i \in I$.
- (2) For $j \in I$, we define

$$\phi^{(j)} := \mathcal{F}^{-1}(\widehat{\phi_j} \circ S_j^{-1}) = |\det T_j| \cdot M_{b_j}[\phi_j \circ T_j^T],$$

so that $\phi^{(j)}$ is to ϕ_j as $\gamma^{(j)}$ is to γ_j .

- (3) For $j, i \in I$, set

$$B_{j,i} := \begin{cases} (1 + \|T_j^{-1}T_i\|)^{K+d} \cdot \|\mathcal{F}^{-1}(\varphi_i \cdot \widehat{\phi^{(j)}})\|_{L_{v_0}^1}, & \text{if } p \in [1, \infty], \\ (1 + \|T_j^{-1}T_i\|)^{pK+d} \cdot |\det T_i|^{1-p} \cdot \|\mathcal{F}^{-1}(\varphi_i \cdot \widehat{\phi^{(j)}})\|_{L_{v_0}^p}^p, & \text{if } p \in (0, 1). \end{cases}$$

- (4) With

$$r := \max \left\{ q, \frac{q}{p} \right\} = \begin{cases} q, & \text{if } p \in [1, \infty], \\ \frac{q}{p}, & \text{if } p \in (0, 1) \end{cases}$$

as in Assumption 3.1, we assume that the operator \vec{B} induced by $(B_{j,i})_{j,i \in I}$, i.e.

$$\vec{B}(c_i)_{i \in I} := \left(\sum_{j \in I} B_{j,i} c_i \right)_{j \in I}$$

defines a well-defined, bounded operator $\vec{B} : \ell_{w^{\min\{1,p\}}}^r(I) \rightarrow \ell_{w^{\min\{1,p\}}}^r(I)$.

- (5) For $j \in I$ and $\delta \in (0, 1]$, we let the j -th **coefficient space** $C_j^{(\delta)}$ be defined as in equation (4.2) and set

$$W_j := \left\{ f : \mathbb{R}^d \rightarrow \mathbb{C} \mid f \text{ continuous and } \|f\|_{W_j} < \infty \right\},$$

where

$$\|f\|_{W_j} := \|f\|_{V_j} + \sup_{0 < \delta \leq 1} \frac{1}{\delta} \left\| \text{osc}_{\delta, T_j^{-T}[-1,1]^d} [M_{-b_j} f] \right\|_{V_j}.$$

- (6) Finally, we define

$$\ell_w^q([W_j]_{j \in I}) := \left\{ (f_j)_{j \in I} \mid (\forall j \in I : f_j \in W_j) \text{ and } (\|f_j\|_{W_j})_{j \in I} \in \ell_w^q(I) \right\},$$

equipped with the natural (quasi)-norm $\|(f_j)_{j \in I}\|_{\ell_w^q([W_j]_{j \in I})} := \left\| (\|f_j\|_{W_j})_{j \in I} \right\|_{\ell_w^q}$. ◀

Remark. Note that $\varphi_i \cdot \widehat{\phi^{(j)}} \in C_c^\infty(\mathbb{R}^d)$ for all $i, j \in I$, since $\varphi_i \in C_c^\infty(\mathbb{R}^d)$ and since $\widehat{\phi^{(j)}} = \widehat{\phi_j} \circ S_j^{-1}$ is smooth, because $\widehat{\phi_j}$ is. Hence, $\mathcal{F}^{-1}(\varphi_i \cdot \widehat{\phi^{(j)}}) \in \mathcal{S}(\mathbb{R}^d)$, so that $B_{j,i} < \infty$ for all $i, j \in I$, cf. Lemma 2.3.

Although we again stated the assumption in the general case where the prototype γ_i may depend on $i \in I$, the reader should keep in mind the most important special case where $\gamma_i = \gamma$ is independent of $i \in I$. ◆

Given these assumptions, we now want to show in particular that $\gamma^{[j]} * f$ is continuous for each $f \in \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$. The following lemma makes an important step in that direction.

Lemma 4.2. Assume that $\Gamma = (\gamma_i)_{i \in I}$ satisfies Assumption 4.1. Then the following hold:

For $p \in [1, \infty]$ and

$$C := \frac{2^{6d}}{\sqrt{d}} \cdot \left(1152 \cdot d^{5/2} \cdot \lceil K + d + 1 \rceil \right)^{\lceil K \rceil + d + 2} \cdot \Omega_0^{2K} \Omega_1^2 \cdot (1 + R_{\mathcal{Q}})^d,$$

we have

$$\left\| \text{osc}_{\delta, T_j^{-T}[-1,1]^d} \left[M_{-b_j} \mathcal{F}^{-1} \left(\widehat{\gamma^{(j)}} \cdot \varphi_i \widehat{f} \right) \right] \right\|_{L_v^p} \leq C \cdot \delta \cdot (1 + \|T_j^{-1} T_i\|)^{K+d} \cdot \|\mathcal{F}^{-1}(\varphi_i \cdot \widehat{\phi^{(j)}})\|_{L_{v_0}^1} \cdot \|\mathcal{F}^{-1}(\varphi_i^* \widehat{f})\|_{L_v^p}$$

for all $0 < \delta \leq 1$, all $f \in Z'(\mathcal{O})$ and all $i, j \in I$.

Likewise, for $p \in (0, 1)$ and

$$C := \frac{2^{16\frac{d}{p}} \cdot (1 + C_{\mathcal{Q}} R_{\mathcal{Q}})^{\frac{2d}{p}}}{370 \cdot d^{11/2} \cdot d^{d/2p}} \cdot \left(4032 \cdot d^3 \cdot \left\lceil K + \frac{d+1}{p} \right\rceil \right)^{2\lceil K + \frac{d+1}{p} \rceil + 2} \cdot \Omega_0^{5K} \Omega_1^5,$$

we have

$$\begin{aligned} & \left\| \text{osc}_{\delta, T_j^{-T}[-1,1]^d} \left[M_{-b_j} \mathcal{F}^{-1} \left(\varphi_i \widehat{\gamma^{(j)}} \widehat{f} \right) \right] \right\|_{W_{T_j^{-T}[-1,1]^d}(L_v^p)} \\ & \leq C \cdot \delta \cdot |\det T_i|^{\frac{1}{p}-1} \cdot (1 + \|T_j^{-1} T_i\|)^{K+\frac{d}{p}} \cdot \|\mathcal{F}^{-1}(\varphi_i \widehat{\phi^{(j)}})\|_{L_{v_0}^p} \cdot \|\mathcal{F}^{-1}(\varphi_i^* \widehat{f})\|_{L_v^p} \end{aligned}$$

for all $0 < \delta \leq 1$, all $f \in Z'(\mathcal{O})$ and all $i, j \in I$. ◀

Proof. First of all, note that $\widehat{f} \in \mathcal{D}'(\mathcal{O})$ for $f \in Z'(\mathcal{O})$. Because of $\varphi_i \in C_c^\infty(\mathcal{O})$, this implies that $\varphi_i \cdot \widehat{f} \in \mathcal{S}'(\mathbb{R}^d)$ is a well-defined tempered distribution with compact support, so that the same holds for $\varphi_i \cdot \widehat{\gamma^{(j)}} \cdot \widehat{f}$, since $\widehat{\gamma^{(j)}} \in C^\infty(\mathbb{R}^d)$. Hence, by the Paley-Wiener theorem, $\mathcal{F}^{-1}(\widehat{\gamma^{(j)}} \cdot \varphi_i \cdot \widehat{f})$ is a smooth (even analytic) function with polynomially bounded derivatives of all orders. In particular, expressions like $\left(\text{osc}_{\delta, T_j^{-T}[-1,1]^d} \left[M_{-b_j} \mathcal{F}^{-1}(\widehat{\gamma^{(j)}} \cdot \varphi_i \widehat{f}) \right] \right)(x)$ are well-defined for every $x \in \mathbb{R}^d$.

Let $f \in Z'(\mathcal{O})$ be arbitrary. We can clearly assume $\|\mathcal{F}^{-1}(\varphi_i^* \cdot \widehat{f})\|_{L_v^p} < \infty$, since otherwise, the claim is trivial. Now, note that $\widehat{\gamma^{(j)}} \cdot \varphi_i \in C_c^\infty(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$ and $\varphi_i^* \widehat{f} \in \mathcal{S}'(\mathbb{R}^d)$, as well as

$$\begin{aligned} M_{-b_j} \left[\mathcal{F}^{-1}(\widehat{\gamma^{(j)}} \cdot \varphi_i \cdot \widehat{f}) \right] &= \mathcal{F}^{-1} \left[L_{-b_j} \left(\widehat{\gamma^{(j)}} \varphi_i \cdot \varphi_i^* \widehat{f} \right) \right] \\ &= \mathcal{F}^{-1} \left[L_{-b_j} \left(\widehat{\gamma^{(j)}} \cdot \varphi_i \right) \right] * \mathcal{F}^{-1} \left[L_{-b_j} (\varphi_i^* \widehat{f}) \right]. \end{aligned}$$

In particular, the convolution is pointwise well-defined, so that Lemma 2.12 shows

$$\text{osc}_{\delta, T_j^{-T}[-1,1]^d} M_{-b_j} \left[\mathcal{F}^{-1}(\widehat{\gamma^{(j)}} \cdot \varphi_i \widehat{f}) \right] \leq \left(\text{osc}_{\delta, T_j^{-T}[-1,1]^d} \mathcal{F}^{-1} \left[L_{-b_j} \left(\widehat{\gamma^{(j)}} \cdot \varphi_i \right) \right] \right) * \left| \mathcal{F}^{-1} \left[L_{-b_j} (\varphi_i^* \widehat{f}) \right] \right|. \quad (4.3)$$

Now, for $p \in (0, 1)$, we want to apply Proposition 2.21 with $Q_1 = T_i^{-T}[-1, 1]^d$, $Q_2 = T_j^{-T}[-1, 1]^d$ and

$$g = \left| \mathcal{F}^{-1} \left[L_{-b_j} (\varphi_i^* \widehat{f}) \right] \right|, \quad \text{as well as} \quad f = \text{osc}_{\delta, T_j^{-T}[-1,1]^d} \mathcal{F}^{-1} \left[L_{-b_j} \left(\widehat{\gamma^{(j)}} \cdot \varphi_i \right) \right].$$

To this end, first note just as in the proof of Corollary 2.22 (cf. equation (2.18)) that with this choice of Q_1 and suitable choices of $(x_i)_{i \in \mathbb{Z}^d}$, the constant N from Theorem 2.19 satisfies $N \leq 3^d$.

Next, we use the identities $Q_1 - Q_1 = T_i^{-T}[-2, 2]^d$ and $|\mathcal{F}^{-1}[L_b h]| = |M_b[\mathcal{F}^{-1}h]| = |\mathcal{F}^{-1}h|$ and equation (2.2), as well as Theorem 2.17 to get

$$\begin{aligned} & \left\| |\mathcal{F}^{-1}[L_{-b_j}(\varphi_i^* \widehat{f})]| \right\|_{W_{Q_1 - Q_1}(L_v^p)} = \|\mathcal{F}^{-1}(\varphi_i^* \widehat{f})\|_{W_{T_i^{-T}[-2,2]^d}(L_v^p)} \\ & \quad (\text{eq. (2.2)}) \leq \Omega_0^K \Omega_1 \cdot (18d)^{K+\frac{d}{p}} \cdot \|\mathcal{F}^{-1}(\varphi_i^* \widehat{f})\|_{W_{T_i^{-T}[-1,1]^d}(L_v^p)} \\ & \quad (\text{Theorem 2.17}) \leq \Omega_0^K \Omega_1 \cdot (18d)^{K+\frac{d}{p}} C_1 \cdot \|\mathcal{F}^{-1}(\varphi_i^* \widehat{f})\|_{L_v^p} \end{aligned}$$

for

$$C_1 := 2^{4(1+\frac{d}{p})} s_d^{\frac{1}{p}} \left(192 \cdot d^{3/2} \cdot \left\lceil K + \frac{d+1}{p} \right\rceil \right)^{\lceil K + \frac{d+1}{p} \rceil + 1} \cdot \Omega_0^K \Omega_1 \cdot (1 + (1 + 2C_{\mathcal{Q}}) R_{\mathcal{Q}})^{\frac{d}{p}},$$

since [77, Lemma 2.7] shows $\text{supp}(\varphi_i^* \widehat{f}) \subset \overline{Q_i^*} \subset T_i[\overline{B_R}(0)] + b_i \subset T_i[-R, R]^d + b_i$ for $R = (1 + 2C_{\mathcal{Q}}) R_{\mathcal{Q}}$.

All in all, we now set $C_2 := \Omega_0^K \Omega_1 \cdot (18d)^{K+\frac{d}{p}} C_1$ and use equation (4.3), Proposition 2.21, and the identity $\widehat{\gamma^{(j)}} = L_{b_j}(\widehat{\gamma_j} \circ T_j^{-1})$ to conclude because of

$$\begin{aligned} \sup_{x \in Q_1} v_0(x) &\leq \Omega_1 \cdot \sup_{x \in T_i^{-T}[-1,1]^d} (1+|x|)^K \\ (\text{eq. (1.11)}) &\leq \Omega_0^K \Omega_1 \cdot \sup_{x \in T_i^{-T}[-1,1]^d} (1+|T_i^T x|)^K \\ &= \Omega_0^K \Omega_1 \cdot \sup_{y \in [-1,1]^d} (1+|y|)^K \\ &\leq \Omega_0^K \Omega_1 (1+\sqrt{d})^K \end{aligned}$$

that

$$\begin{aligned} &\left\| \text{osc}_{\delta, T_j^{-T}[-1,1]^d} \left[M_{-b_j} \mathcal{F}^{-1} \left(\widehat{\gamma^{(j)}} \cdot \varphi_i \widehat{f} \right) \right] \right\|_{W_{T_j^{-T}[-1,1]^d}(L_{v_0}^p)} \\ &\leq 3^{\frac{d}{p}} \Omega_0^K \Omega_1 (1+\sqrt{d})^K \cdot [\lambda_d(Q_1)]^{1-\frac{1}{p}} \cdot \left\| \mathcal{F}^{-1} [L_{-b_j}(\varphi_i^* \widehat{f})] \right\|_{W_{Q_1-Q_1}(L_{v_0}^p)} \\ &\quad \cdot \left\| \text{osc}_{\delta, T_j^{-T}[-1,1]^d} \left(\mathcal{F}^{-1} [L_{-b_j}(\widehat{\gamma^{(j)}} \cdot \varphi_i)] \right) \right\|_{W_{T_j^{-T}[-1,1]^d - T_i^{-T}[-1,1]^d}(L_{v_0}^p)} \\ &\leq 3^{\frac{d}{p}} \Omega_0^K \Omega_1 (1+\sqrt{d})^K C_2 \cdot 2^{d(1-\frac{1}{p})} |\det T_i|^{\frac{1}{p}-1} \cdot \left\| \mathcal{F}^{-1}(\varphi_i^* \widehat{f}) \right\|_{L_{v_0}^p} \\ &\quad \cdot \left\| \text{osc}_{\delta, T_j^{-T}[-1,1]^d} \left(\mathcal{F}^{-1} [L_{-b_j}(L_{b_j}(\widehat{\gamma_j} \circ T_j^{-1}) \cdot \varphi_i)] \right) \right\|_{W_{T_j^{-T}[-1,1]^d - T_i^{-T}[-1,1]^d}(L_{v_0}^p)} \\ &\leq (2\sqrt{d})^K 3^{\frac{d}{p}} C_2 \cdot \Omega_0^K \Omega_1 \cdot |\det T_i|^{\frac{1}{p}-1} \cdot \left\| \mathcal{F}^{-1}(\varphi_i^* \widehat{f}) \right\|_{L_{v_0}^p} \\ &\quad \cdot \left\| \text{osc}_{\delta, T_j^{-T}[-1,1]^d} \left(\mathcal{F}^{-1} [(\widehat{\gamma_j} \cdot [(L_{-b_j} \varphi_i) \circ T_j]) \circ T_j^{-1}] \right) \right\|_{W_{T_j^{-T}[-1,1]^d - T_i^{-T}[-1,1]^d}(L_{v_0}^p)}. \end{aligned}$$

Now, we recall $\phi_j = \nabla \gamma_j$ and estimate

$$\begin{aligned} &\left\| \text{osc}_{\delta, T_j^{-T}[-1,1]^d} \left(\mathcal{F}^{-1} [(\widehat{\gamma_j} \cdot [(L_{-b_j} \varphi_i) \circ T_j]) \circ T_j^{-1}] \right) \right\|_{W_{T_j^{-T}[-1,1]^d - T_i^{-T}[-1,1]^d}(L_{v_0}^p)} \\ &= |\det T_j| \cdot \left\| \text{osc}_{\delta, T_j^{-T}[-1,1]^d} \left[(\mathcal{F}^{-1} [\widehat{\gamma_j} \cdot ((L_{-b_j} \varphi_i) \circ T_j)]) \circ T_j^T \right] \right\|_{W_{T_j^{-T}[-1,1]^d - T_i^{-T}[-1,1]^d}(L_{v_0}^p)} \\ (\text{Lem. 2.11, 2.4}) &= |\det T_j| \cdot \left\| \left(M_{[-1,1]^d - T_j^T T_i^{-T}[-1,1]^d} \left[\text{osc}_{\delta, [-1,1]^d} \left(\mathcal{F}^{-1} [\widehat{\gamma_j} \cdot ((L_{-b_j} \varphi_i) \circ T_j)] \right) \right] \right) \circ T_j^T \right\|_{L_{v_0}^p} \\ &= |\det T_j|^{1-\frac{1}{p}} \left\| (v_0 \circ T_j^{-T}) \cdot M_{[-1,1]^d - T_j^T T_i^{-T}[-1,1]^d} \left[\text{osc}_{\delta, [-1,1]^d} \left(\mathcal{F}^{-1} [\widehat{\gamma_j} \cdot ((L_{-b_j} \varphi_i) \circ T_j)] \right) \right] \right\|_{L^p} \\ (\text{Lem. 2.13}) &\leq 2\sqrt{d} \cdot \delta \cdot |\det T_j|^{1-\frac{1}{p}} \cdot \left\| (v_0 \circ T_j^{-T}) \cdot M_{[-1,1]^d - T_j^T T_i^{-T}[-1,1]^d} \left[M_{\delta, [-1,1]^d} \nabla \left(\mathcal{F}^{-1} [\widehat{\gamma_j} \cdot ((L_{-b_j} \varphi_i) \circ T_j)] \right) \right] \right\|_{L^p}; \end{aligned}$$

Since we have $\delta \leq 1$, Lemma 2.5 allows us to continue the estimate as follows:

$$\begin{aligned} &\dots \leq 2\sqrt{d} \cdot \delta \cdot |\det T_j|^{1-\frac{1}{p}} \cdot \left\| (v_0 \circ T_j^{-T}) \cdot M_{[-2,2]^d - T_j^T T_i^{-T}[-1,1]^d} \left[\nabla (\gamma_j * \mathcal{F}^{-1} [(L_{-b_j} \varphi_i) \circ T_j]) \right] \right\|_{L^p} \\ (\nabla(f*g) = (\nabla f)*g) &\stackrel{(*)}{=} 2\sqrt{d} \cdot \delta \cdot |\det T_j|^{1-\frac{1}{p}} \cdot \left\| (v_0 \circ T_j^{-T}) \cdot M_{[-2,2]^d - T_j^T T_i^{-T}[-1,1]^d} (\phi_j * \mathcal{F}^{-1} [(L_{-b_j} \varphi_i) \circ T_j]) \right\|_{L^p} \\ &= 2\sqrt{d} \cdot \delta \cdot |\det T_j|^{1-\frac{1}{p}} \cdot \left\| (v_0 \circ T_j^{-T}) \cdot M_{[-2,2]^d - T_j^T T_i^{-T}[-1,1]^d} \left[\mathcal{F}^{-1} \left(\left(\widehat{\phi_j} \circ T_j^{-1} \right) \cdot (L_{-b_j} \varphi_i) \right) \circ T_j \right] \right\|_{L^p} \\ &= 2\sqrt{d} \cdot \delta \cdot |\det T_j|^{1-\frac{1}{p}} \cdot \left\| (v_0 \circ T_j^{-T}) \cdot M_{[-2,2]^d - T_j^T T_i^{-T}[-1,1]^d} \left[\left(\mathcal{F}^{-1} \left(\left(\widehat{\phi_j} \circ T_j^{-1} \right) \cdot (L_{-b_j} \varphi_i) \right) \right) \circ T_j^{-T} \right] \right\|_{L^p} \\ (\text{Lemma 2.4}) &= 2\sqrt{d} \cdot \delta \cdot |\det T_j|^{1-\frac{1}{p}} \cdot \left\| \left[v_0 \cdot M_{T_j^{-T}[-2,2]^d - T_i^{-T}[-1,1]^d} \left(\mathcal{F}^{-1} \left[\left(\widehat{\phi_j} \circ T_j^{-1} \right) \cdot (L_{-b_j} \varphi_i) \right] \right) \right] \circ T_j^{-T} \right\|_{L^p} \\ &= 2\sqrt{d} \cdot \delta \cdot \left\| \mathcal{F}^{-1} \left[\left(\widehat{\phi_j} \circ T_j^{-1} \right) \cdot (L_{-b_j} \varphi_i) \right] \right\|_{W_{T_j^{-T}[-2,2]^d - T_i^{-T}[-1,1]^d}(L_{v_0}^p)} \end{aligned} \tag{4.4}$$

Here, the step marked with $(*)$ is justified, since $\mathcal{F}^{-1}[(L_{-b_j}\varphi_i) \circ T_j] \in \mathcal{S}(\mathbb{R}^d)$ and since $\gamma_j \in L^1(\mathbb{R}^d) \cap C^1(\mathbb{R}^d)$, where $\phi_j = \nabla \gamma_j$ is bounded by Assumptions 3.1 and 4.1.

Next, we observe

$$\begin{aligned} T_j^{-T}[-2, 2]^d - T_i^{-T}[-1, 1]^d &= T_i^{-T} \left(T_i^T T_j^{-T}[-2, 2]^d - [-1, 1]^d \right) \\ &\subset T_i^{-T} \left(2 \left\| (T_j^{-1} T_i)^T \right\|_{\ell^\infty \rightarrow \ell^\infty} [-1, 1]^d - [-1, 1]^d \right) \\ &\subset T_i^{-T} \left[- \left(1 + 2 \left\| T_j^{-1} T_i \right\|_{\ell^1 \rightarrow \ell^1} \right), 1 + 2 \left\| T_j^{-1} T_i \right\|_{\ell^1 \rightarrow \ell^1} \right]^d. \end{aligned}$$

Consequently, if we set $R := 1 + 2 \left\| T_j^{-1} T_i \right\|_{\ell^1 \rightarrow \ell^1}$ for brevity, then Corollary 2.9 (with v_0 instead of v , with $i = j$ and with $L = 1$) yields for arbitrary measurable $h : \mathbb{R}^d \rightarrow \mathbb{C}^k$ the estimate

$$\begin{aligned} \|h\|_{W_{T_j^{-T}[-2, 2]^d - T_i^{-T}[-1, 1]^d}^k(L_{v_0}^p)} &\leq \|h\|_{W_{T_i^{-T}[-R, R]^d}^k(L_{v_0}^p)} \\ &\leq \Omega_0^K \Omega_1 \cdot \left[3d \left(1 + 1 + 2 \left\| T_j^{-1} T_i \right\|_{\ell^1 \rightarrow \ell^1} \right) \right]^{K + \frac{d}{p}} \cdot (1 + 1)^{K + \frac{d}{p}} \cdot \|h\|_{W_{T_i^{-T}[-1, 1]^d}^k(L_{v_0}^p)} \\ &\quad (\text{since } \|A\|_{\ell^1 \rightarrow \ell^1} \leq \sqrt{d} \|A\|) \leq \Omega_0^K \Omega_1 \cdot \left[12 \cdot d^{\frac{3}{2}} (1 + \left\| T_j^{-1} T_i \right\|) \right]^{K + \frac{d}{p}} \cdot \|h\|_{W_{T_i^{-T}[-1, 1]^d}^k(L_{v_0}^p)}. \end{aligned}$$

Now, we use this estimate and standard properties of the Fourier transform to further estimate the right-hand side of equation (4.4) as follows:

$$\begin{aligned} \text{r.h.s.}(4.4) &= 2\sqrt{d} \cdot \delta \cdot \left\| \mathcal{F}^{-1} \left(L_{-b_j} \left[\varphi_i \cdot L_{b_j} \left(\widehat{\phi_j} \circ T_j^{-1} \right) \right] \right) \right\|_{W_{T_j^{-T}[-2, 2]^d - T_i^{-T}[-1, 1]^d}^d(L_{v_0}^p)} \\ &= 2\sqrt{d} \cdot \delta \cdot \left\| \mathcal{F}^{-1} \left(L_{-b_j} \left[\varphi_i \cdot \widehat{\phi^{(j)}} \right] \right) \right\|_{W_{T_j^{-T}[-2, 2]^d - T_i^{-T}[-1, 1]^d}^d(L_{v_0}^p)} \\ &= 2\sqrt{d} \cdot \delta \cdot \left\| \mathcal{F}^{-1} \left[\widehat{\phi^{(j)}} \cdot \varphi_i \right] \right\|_{W_{T_j^{-T}[-2, 2]^d - T_i^{-T}[-1, 1]^d}^d(L_{v_0}^p)} \\ &\leq 2\sqrt{d} \cdot \Omega_0^K \Omega_1 \cdot \left[12 \cdot d^{\frac{3}{2}} (1 + \left\| T_j^{-1} T_i \right\|) \right]^{K + \frac{d}{p}} \cdot \delta \cdot \left\| \mathcal{F}^{-1} \left[\widehat{\phi^{(j)}} \cdot \varphi_i \right] \right\|_{W_{T_i^{-T}[-1, 1]^d}^d(L_{v_0}^p)}. \quad (4.5) \end{aligned}$$

Recall that we are in the case $p \in (0, 1)$. In particular, we have $|y| \leq \|y\|_{\ell^p}$ for each $y \in \mathbb{R}^d$. For a vector-valued function $f : \mathbb{R}^d \rightarrow \mathbb{R}^k$ and any (measurable) weight $u : \mathbb{R}^d \rightarrow (0, \infty)$, this implies

$$\begin{aligned} \|f\|_{W_Q^k(L_u^p)}^p &= \int_{\mathbb{R}^d} [u(x) \cdot |(M_Q f)(x)|]^p dx \\ &= \int_{\mathbb{R}^d} [u(x)]^p \cdot \text{ess sup}_{y \in x+Q} |f(y)|^p dx \\ &\leq \int_{\mathbb{R}^d} [u(x)]^p \cdot \text{ess sup}_{y \in x+Q} \|f(y)\|_{\ell^p}^p dx \\ &= \int_{\mathbb{R}^d} [u(x)]^p \cdot \text{ess sup}_{y \in x+Q} \sum_{\ell=1}^k |f_\ell(y)|^p dx \\ &\leq \sum_{\ell=1}^k \int_{\mathbb{R}^d} [u(x)]^p \cdot \text{ess sup}_{y \in x+Q} |f_\ell(y)|^p dx \\ &\leq k \cdot \max_{\ell \in \underline{k}} \|f_\ell\|_{W_Q(L_u^p)}^p. \end{aligned}$$

In other words, we have shown

$$\|f\|_{W_Q^k(L_u^p)} \leq k^{1/p} \cdot \max_{\ell \in \underline{k}} \|f_\ell\|_{W_Q(L_u^p)}. \quad (4.6)$$

Using this inequality (with $k = d$), we can further estimate the right-hand side of equation (4.5) as follows:

$$\begin{aligned}
\text{r.h.s.}(4.5) &\leq 2d^{\frac{1}{2}+\frac{1}{p}} \cdot \Omega_0^K \Omega_1 \cdot \left[12 \cdot d^{\frac{3}{2}} (1 + \|T_j^{-1}T_i\|)\right]^{K+\frac{d}{p}} \cdot \delta \cdot \max_{\ell \in \underline{d}} \left\| \mathcal{F}^{-1} \left(\left[\widehat{\phi^{(j)}} \right]_{\ell} \cdot \varphi_i \right) \right\|_{W_{T_i^{-T}[-1,1]^d}(L_{v_0}^p)} \\
&\stackrel{(\text{Theorem 2.17})}{\leq} 2d^{\frac{1}{2}+\frac{1}{p}} \cdot \Omega_0^K \Omega_1 \cdot \left[12 \cdot d^{\frac{3}{2}} (1 + \|T_j^{-1}T_i\|)\right]^{K+\frac{d}{p}} C_3 \cdot \delta \cdot \max_{\ell \in \underline{d}} \left\| \mathcal{F}^{-1} \left(\left[\widehat{\phi^{(j)}} \right]_{\ell} \cdot \varphi_i \right) \right\|_{L_{v_0}^p} \\
&\stackrel{(\text{since } d \leq 2^d)}{\leq} 2d^{\frac{1}{2}} 2^{\frac{d}{p}} \cdot \Omega_0^K \Omega_1 \cdot \left[12 \cdot d^{\frac{3}{2}} (1 + \|T_j^{-1}T_i\|)\right]^{K+\frac{d}{p}} C_3 \cdot \delta \cdot \left\| \mathcal{F}^{-1} \left[\widehat{\phi^{(j)}} \cdot \varphi_i \right] \right\|_{L_{v_0}^p} \\
&=: C_4 \cdot \delta \cdot (1 + \|T_j^{-1}T_i\|)^{K+\frac{d}{p}} \cdot \left\| \mathcal{F}^{-1} \left[\widehat{\phi^{(j)}} \cdot \varphi_i \right] \right\|_{L_{v_0}^p}.
\end{aligned}$$

Here,

$$C_3 = 2^{4(1+\frac{d}{p})} s_d^{\frac{1}{p}} \left(192 \cdot d^{3/2} \cdot \left\lceil K + \frac{d+1}{p} \right\rceil \right)^{\lceil K + \frac{d+1}{p} \rceil + 1} \cdot \Omega_0^K \Omega_1 \cdot (1 + R_{\mathcal{Q}})^{\frac{d}{p}},$$

cf. Theorem 2.17, since we have for arbitrary $\ell \in \underline{d}$ that

$$\text{supp} \left(\left[\widehat{\phi^{(j)}} \right]_{\ell} \cdot \varphi_i \right) \subset \overline{Q_i} \subset T_i [\overline{B_{R_{\mathcal{Q}}}}(0)] + b_i \subset T_i [-R_{\mathcal{Q}}, R_{\mathcal{Q}}]^d + b_i.$$

Putting everything together, we arrive at

$$\begin{aligned}
&\left\| \text{osc}_{\delta, T_j^{-T}[-1,1]^d} \left[M_{-b_j} \mathcal{F}^{-1} \left(\widehat{\gamma^{(j)}} \cdot \varphi_i \widehat{f} \right) \right] \right\|_{W_{T_j^{-T}[-1,1]^d}(L_v^p)} \\
&\leq \left(2\sqrt{d} \right)^K 3^{\frac{d}{p}} C_2 \Omega_0^K \Omega_1 \cdot |\det T_i|^{\frac{1}{p}-1} \cdot \left\| \mathcal{F}^{-1}(\varphi_i^* \widehat{f}) \right\|_{L_v^p} \\
&\quad \cdot \left\| \text{osc}_{\delta, T_j^{-T}[-1,1]^d} \left(\mathcal{F}^{-1} \left[(\widehat{\gamma_j} \cdot [(L_{-b_j} \varphi_i) \circ T_j]) \circ T_j^{-1} \right] \right) \right\|_{W_{T_j^{-T}[-1,1]^d - T_i^{-T}[-1,1]^d}(L_{v_0}^p)} \\
&\leq \left(2\sqrt{d} \right)^K 3^{\frac{d}{p}} C_2 C_4 \cdot \Omega_0^K \Omega_1 \cdot |\det T_i|^{\frac{1}{p}-1} \cdot \delta \cdot (1 + \|T_j^{-1}T_i\|)^{K+\frac{d}{p}} \cdot \left\| \mathcal{F}^{-1}(\varphi_i^* \widehat{f}) \right\|_{L_v^p} \cdot \left\| \mathcal{F}^{-1} \left[\widehat{\phi^{(j)}} \cdot \varphi_i \right] \right\|_{L_{v_0}^p}.
\end{aligned}$$

This establishes the claim for $p \in (0, 1)$, since we have $C_{\mathcal{Q}} \geq \|T_i^{-1}T_i\| \geq 1$ and $s_d \leq 2^{2d}$ and hence

$$\begin{aligned}
&\left(2\sqrt{d} \right)^K 3^{\frac{d}{p}} C_2 C_4 \cdot \Omega_0^K \Omega_1 \\
&= C_1 \cdot 2^5 d^{1/2} \cdot \left(2\sqrt{d} \right)^K 96^{\frac{d}{p}} \cdot \left(216 \cdot d^{\frac{5}{2}} \right)^{K+\frac{d}{p}} \cdot s_d^{\frac{1}{p}} \left(192 \cdot d^{3/2} \cdot \left\lceil K + \frac{d+1}{p} \right\rceil \right)^{\lceil K + \frac{d+1}{p} \rceil + 1} \cdot \Omega_0^{4K} \Omega_1^4 \cdot (1 + R_{\mathcal{Q}})^{\frac{d}{p}} \\
&\leq 2^9 d^{\frac{1}{2}} \cdot 2^{15\frac{d}{p}} \left(2\sqrt{d} \right)^{-\frac{d}{p}} \cdot (432 \cdot d^3)^{K+\frac{d}{p}} \cdot \left(192 \cdot d^{3/2} \cdot \left\lceil K + \frac{d+1}{p} \right\rceil \right)^{2\lceil K + \frac{d+1}{p} \rceil + 2} \cdot \Omega_0^{5K} \Omega_1^5 \cdot (1 + R_{\mathcal{Q}})^{\frac{d}{p}} (1 + 3C_{\mathcal{Q}}R_{\mathcal{Q}})^{\frac{d}{p}} \\
&\leq 2^9 d^{1/2} \cdot 2^{17\frac{d}{p}} \left(2\sqrt{d} \right)^{-\frac{d}{p}} \cdot (21 \cdot d^{3/2})^{-4} \cdot \left(4032 \cdot d^3 \cdot \left\lceil K + \frac{d+1}{p} \right\rceil \right)^{2\lceil K + \frac{d+1}{p} \rceil + 2} \cdot \Omega_0^{5K} \Omega_1^5 \cdot (1 + C_{\mathcal{Q}}R_{\mathcal{Q}})^{\frac{2d}{p}} \\
&\leq \frac{2^{16\frac{d}{p}} \cdot (1 + C_{\mathcal{Q}}R_{\mathcal{Q}})^{\frac{2d}{p}}}{370 \cdot d^{11/2} \cdot d^{d/2p}} \cdot \left(4032 \cdot d^3 \cdot \left\lceil K + \frac{d+1}{p} \right\rceil \right)^{2\lceil K + \frac{d+1}{p} \rceil + 2} \cdot \Omega_0^{5K} \Omega_1^5.
\end{aligned}$$

For $p \in [1, \infty]$, the proof is simpler: We use the weighted Young inequality (equation (1.12)) and equation (4.3) to derive

$$\begin{aligned}
& \left\| \text{osc}_{\delta \cdot T_j^{-T}[-1,1]^d} \left(M_{-b_j} \left[\mathcal{F}^{-1} \left(\widehat{\gamma^{(j)}} \cdot \varphi_i \widehat{f} \right) \right] \right) \right\|_{L_v^p} \\
& \stackrel{(\text{eqs. (4.3), (1.12)})}{\leq} \left\| \text{osc}_{\delta \cdot T_j^{-T}[-1,1]^d} \mathcal{F}^{-1} \left[L_{-b_j} \left(\widehat{\gamma^{(j)}} \cdot \varphi_i \right) \right] \right\|_{L_{v_0}^1} \cdot \left\| \mathcal{F}^{-1} [L_{-b_j} (\varphi_i^* \widehat{f})] \right\|_{L_v^p} \\
& \left(\widehat{\gamma^{(j)}} = L_{b_j} (\widehat{\gamma_j} \circ T_j^{-1}) \right) = \left\| \text{osc}_{\delta \cdot T_j^{-T}[-1,1]^d} \mathcal{F}^{-1} \left[(\widehat{\gamma_j} \circ T_j^{-1}) \cdot (L_{-b_j} \varphi_i) \right] \right\|_{L_{v_0}^1} \cdot \left\| \mathcal{F}^{-1} [L_{-b_j} (\varphi_i^* \widehat{f})] \right\|_{L_v^p} \\
& (|\mathcal{F}^{-1}[L_b h]| = |\mathcal{F}^{-1} h|) = \left\| \text{osc}_{\delta \cdot T_j^{-T}[-1,1]^d} \mathcal{F}^{-1} \left([\widehat{\gamma_j} \cdot ((L_{-b_j} \varphi_i) \circ T_j)] \circ T_j^{-1} \right) \right\|_{L_{v_0}^1} \cdot \left\| \mathcal{F}^{-1} (\varphi_i^* \widehat{f}) \right\|_{L_v^p} \\
& = |\det T_j| \cdot \left\| \text{osc}_{\delta \cdot T_j^{-T}[-1,1]^d} \left[(\mathcal{F}^{-1} [\widehat{\gamma_j} \cdot ((L_{-b_j} \varphi_i) \circ T_j)]) \circ T_j^T \right] \right\|_{L_{v_0}^1} \cdot \left\| \mathcal{F}^{-1} (\varphi_i^* \widehat{f}) \right\|_{L_v^p} \\
& (\text{Lemma 2.11}) = |\det T_j| \cdot \left\| \left(\text{osc}_{\delta \cdot [-1,1]^d} \mathcal{F}^{-1} [\widehat{\gamma_j} \cdot ((L_{-b_j} \varphi_i) \circ T_j)] \right) \circ T_j^T \right\|_{L_{v_0}^1} \cdot \left\| \mathcal{F}^{-1} (\varphi_i^* \widehat{f}) \right\|_{L_v^p} \\
& = \left\| (v_0 \circ T_j^{-T}) \cdot \text{osc}_{\delta \cdot [-1,1]^d} [\gamma_j * \mathcal{F}^{-1} ((L_{-b_j} \varphi_i) \circ T_j)] \right\|_{L^1} \cdot \left\| \mathcal{F}^{-1} (\varphi_i^* \widehat{f}) \right\|_{L_v^p} \\
& (\text{Lemma 2.13}) \leq 2\delta\sqrt{d} \cdot \left\| (v_0 \circ T_j^{-T}) \cdot M_{\delta[-1,1]^d} (\nabla [\gamma_j * \mathcal{F}^{-1} ((L_{-b_j} \varphi_i) \circ T_j)]) \right\|_{L^1} \cdot \left\| \mathcal{F}^{-1} (\varphi_i^* \widehat{f}) \right\|_{L_v^p} \\
& (\text{since } \delta \leq 1) \leq 2\delta\sqrt{d} \cdot \left\| (v_0 \circ T_j^{-T}) \cdot M_{[-1,1]^d} (\nabla [\gamma_j * \mathcal{F}^{-1} ((L_{-b_j} \varphi_i) \circ T_j)]) \right\|_{L^1} \cdot \left\| \mathcal{F}^{-1} (\varphi_i^* \widehat{f}) \right\|_{L_v^p} \\
& (\nabla(\gamma * h) = (\nabla \gamma) * h) = 2\delta\sqrt{d} \cdot \left\| (v_0 \circ T_j^{-T}) \cdot M_{[-1,1]^d} [(\nabla \gamma_j) * (\mathcal{F}^{-1} ((L_{-b_j} \varphi_i) \circ T_j))] \right\|_{L^1} \cdot \left\| \mathcal{F}^{-1} (\varphi_i^* \widehat{f}) \right\|_{L_v^p}.
\end{aligned}$$

Here, the last step is justified just as for $p \in (0, 1)$. Now, we recall $\phi_j = \nabla \gamma_j$ and continue our estimate:

$$\begin{aligned}
& \dots = 2\delta\sqrt{d} \cdot \left\| (v_0 \circ T_j^{-T}) \cdot M_{[-1,1]^d} \left[\mathcal{F}^{-1} \left(\widehat{\phi_j} \cdot ((L_{-b_j} \varphi_i) \circ T_j) \right) \right] \right\|_{L^1} \cdot \left\| \mathcal{F}^{-1} (\varphi_i^* \widehat{f}) \right\|_{L_v^p} \\
& = 2\delta\sqrt{d} \cdot \left\| (v_0 \circ T_j^{-T}) \cdot M_{[-1,1]^d} \left[\mathcal{F}^{-1} \left(\left(\widehat{\phi_j} \circ T_j^{-1} \right) \cdot (L_{-b_j} \varphi_i) \right) \circ T_j \right] \right\|_{L^1} \cdot \left\| \mathcal{F}^{-1} (\varphi_i^* \widehat{f}) \right\|_{L_v^p} \\
& = 2\delta\sqrt{d} \cdot |\det T_j|^{-1} \left\| (v_0 \circ T_j^{-T}) \cdot M_{[-1,1]^d} \left[\left(\mathcal{F}^{-1} [L_{-b_j} [\varphi_i \cdot L_{b_j} (\widehat{\phi_j} \circ T_j^{-1})]] \right) \circ T_j^{-T} \right] \right\|_{L^1} \cdot \left\| \mathcal{F}^{-1} (\varphi_i^* \widehat{f}) \right\|_{L_v^p} \\
& (\text{Lem. 2.4}) = 2\delta\sqrt{d} \cdot |\det T_j|^{-1} \left\| \left[v_0 \cdot M_{T_j^{-T}[-1,1]^d} \left(\mathcal{F}^{-1} [L_{-b_j} [\varphi_i \cdot L_{b_j} (\widehat{\phi_j} \circ T_j^{-1})]] \right) \right] \circ T_j^{-T} \right\|_{L^1} \cdot \left\| \mathcal{F}^{-1} (\varphi_i^* \widehat{f}) \right\|_{L_v^p} \\
& = 2\delta\sqrt{d} \cdot \left\| \mathcal{F}^{-1} (L_{-b_j} [\varphi_i \cdot \widehat{\phi^{(j)}}]) \right\|_{W_{T_j^{-T}[-1,1]^d}^d(L_{v_0}^1)} \cdot \left\| \mathcal{F}^{-1} (\varphi_i^* \widehat{f}) \right\|_{L_v^p} \\
& = 2\delta\sqrt{d} \cdot \left\| \mathcal{F}^{-1} [\varphi_i \cdot \widehat{\phi^{(j)}}] \right\|_{W_{T_j^{-T}[-1,1]^d}^d(L_{v_0}^1)} \cdot \left\| \mathcal{F}^{-1} (\varphi_i^* \widehat{f}) \right\|_{L_v^p}.
\end{aligned}$$

Here, the last step used that $|\mathcal{F}^{-1}[L_b h]| = |\mathcal{F}^{-1} h|$.

Now, we need an analog of equation (4.6) for the case $p \in [1, \infty]$. But for an arbitrary (measurable) weight $u : \mathbb{R}^d \rightarrow (0, \infty)$ and any $q \in [1, \infty]$, the solidity of $W_Q(L_u^q)$ and the triangle inequality for the associated norm yield for any measurable vector-valued function $f = (f_1, \dots, f_k) : \mathbb{R}^d \rightarrow \mathbb{C}^k$ that

$$\|f\|_{W_Q^k(L_u^q)} = \| |f| \|_{W_Q(L_u^q)} \leq \left\| \sum_{\ell=1}^k |f_\ell| \right\|_{W_Q(L_u^q)} \leq \sum_{\ell=1}^k \|f_\ell\|_{W_Q(L_u^q)} \leq k \cdot \max_{\ell \in \underline{k}} \|f_\ell\|_{W_Q(L_u^q)}. \quad (4.7)$$

We now use this estimate (with $k = d$), as well as equation (2.3) and Theorem 2.17 (both with v_0 instead of v) to conclude

$$\begin{aligned}
& \left\| \mathcal{F}^{-1} [\varphi_i \cdot \widehat{\phi^{(j)}}] \right\|_{W_{T_j^{-T}[-1,1]^d}^d(L_{v_0}^1)} \leq \Omega_0^K \Omega_1 \cdot (6d)^{K+d} \cdot (1 + \|T_j^{-1} T_i\|)^{K+d} \cdot \left\| \mathcal{F}^{-1} [\varphi_i \cdot \widehat{\phi^{(j)}}] \right\|_{W_{T_i^{-T}[-1,1]^d}^d(L_{v_0}^1)} \\
& \leq d \cdot \Omega_0^K \Omega_1 \cdot (6d)^{K+d} \cdot (1 + \|T_j^{-1} T_i\|)^{K+d} \cdot \max_{\ell \in \underline{d}} \left\| \mathcal{F}^{-1} [\varphi_i \cdot (\widehat{\phi^{(j)}})_\ell] \right\|_{W_{T_i^{-T}[-1,1]^d}^d(L_{v_0}^1)} \\
& (\text{Thm. 2.17}) \leq C_5 \cdot d \cdot \Omega_0^K \Omega_1 \cdot (6d)^{K+d} \cdot (1 + \|T_j^{-1} T_i\|)^{K+d} \cdot \left\| \mathcal{F}^{-1} [\varphi_i \cdot \widehat{\phi^{(j)}}] \right\|_{L_{v_0}^1}.
\end{aligned}$$

Here, Theorem 2.17 is applicable, since we have $\text{supp} \left(\varphi_i \cdot \left(\widehat{\phi^{(j)}} \right)_\ell \right) \subset \overline{Q_i} \subset T_i [-R_Q, R_Q]^d + b_i$. Hence, that theorem justifies the last step in the estimate above, with constant

$$C_5 := 2^{4(1+d)} s_d \left(192 \cdot d^{3/2} \cdot [K + d + 1] \right)^{[K+d+1]+1} \cdot \Omega_0^K \Omega_1 \cdot (1 + R_Q)^d.$$

It is not hard to see that this implies the claim for $p \in [1, \infty]$. \square

Next, we show that the map Ana_Γ considered in Theorem 3.5 is not merely bounded as a map into $\ell_w^q \left([V_j]_{j \in I} \right)$, but even as a map into the smaller space $\ell_w^q \left([W_j]_{j \in I} \right)$. In particular, this establishes continuity of $\gamma^{(j)} * f$ for every $j \in I$ and arbitrary $f \in \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$.

Lemma 4.3. *Let $p, q \in (0, \infty]$ and assume that $\Gamma = (\gamma_i)_{i \in I}$ fulfills Assumption 4.1.*

Then, the map

$$\text{Ana}_{\text{osc}} : \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q) \rightarrow \ell_w^q \left([W_j]_{j \in I} \right), f \mapsto \left(\gamma^{(j)} * f \right)_{j \in I}$$

is well-defined and bounded, with

$$\| \text{Ana}_{\text{osc}} \| \leq C \cdot 2^{\max\{0, \frac{1}{q}-1\}} \| \Gamma_Q \| \cdot \left(\| \vec{A} \|_{\max\{1, \frac{1}{p}\}} + \| \vec{B} \|_{\max\{1, \frac{1}{p}\}} \right),$$

where $\Gamma_Q : \ell_w^q(I) \rightarrow \ell_w^q(I)$, $c \mapsto c^*$ denotes the \mathcal{Q} -clustering map, i.e., $c_i^* = \sum_{\ell \in i^*} c_\ell$ and where

$$C := \begin{cases} N_Q^{\frac{1}{p}-1} \cdot \frac{2^{16\frac{d}{p}} \cdot (1+C_Q R_Q)^{\frac{2d}{p}}}{370 \cdot d^{11/2} \cdot d^{d/2p}} \cdot \left(4032 \cdot d^3 \cdot [K + \frac{d+1}{p}] \right)^{2[K + \frac{d+1}{p}]+2} \cdot \Omega_0^{5K} \Omega_1^5, & \text{if } p \in (0, 1), \\ \frac{2^{6d}}{\sqrt{d}} \cdot (1152 \cdot d^{5/2} \cdot [K + d + 1])^{[K]+d+2} \cdot \Omega_0^{2K} \Omega_1^2 \cdot (1 + R_Q)^d, & \text{if } p \in [1, \infty]. \end{cases}$$

Furthermore, we have

$$\left(\gamma^{(j)} * f \right)(x) = \sum_{i \in I} \mathcal{F}^{-1} \left(\widehat{\gamma^{(j)}} \cdot \varphi_i \cdot \widehat{f} \right)(x) \quad \forall x \in \mathbb{R}^d \quad \forall f \in \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q), \quad (4.8)$$

with locally uniform convergence of the series. \blacktriangleleft

Proof. Recall from Theorem 3.5 and from the ensuing remark (which contains the definition of $\gamma^{(j)} * f$) that

$$\left(\gamma^{(j)} * f \right)(x) = \sum_{i \in I} \mathcal{F}^{-1} \left(\widehat{\gamma^{(j)}} \cdot \varphi_i \cdot \widehat{f} \right)(x) \quad \forall f \in \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q), \quad (4.9)$$

where we already know that the series converges absolutely almost everywhere. Next, note that each of the summands of the series above is a smooth function; this follows from the Paley-Wiener theorem, since $\widehat{\gamma^{(j)}} \cdot \varphi_i \cdot \widehat{f}$ is a (tempered) distribution with compact support. Thus, to prove continuity of $\gamma^{(j)} * f$, it suffices to show that the series actually converges *locally* uniformly; by continuity of the summands, for this it suffices to have convergence in $L_{(1+|\bullet|)^{-K}}^\infty(\mathbb{R}^d)$. We will prove this convergence in $L_{(1+|\bullet|)^{-K}}^\infty(\mathbb{R}^d)$ simultaneously with the boundedness of Ana_{osc} .

Let us first consider the case $p \in [1, \infty]$. Here, we let $C_1 > 0$ be the constant provided by Lemma 4.2 (for $p \in [1, \infty]$), so that we get for arbitrary $0 < \delta \leq 1$ the estimate

$$\begin{aligned} & \frac{1}{\delta} \sum_{i \in I} \left\| \text{osc}_{\delta \cdot T_j^{-T}[-1,1]^d} \left(M_{-b_j} \left[\mathcal{F}^{-1} \left(\widehat{\gamma^{(j)}} \cdot \varphi_i \cdot \widehat{f} \right) \right] \right) \right\|_{L_v^p} \\ & \stackrel{(\text{Lemma 4.2})}{\leq} C_1 \cdot \sum_{i \in I} \left[(1 + \|T_j^{-1} T_i\|)^{K+d} \cdot \left\| \mathcal{F}^{-1} \left[\varphi_i \cdot \widehat{\phi^{(j)}} \right] \right\|_{L_{v_0}^1} \cdot \left\| \mathcal{F}^{-1}(\varphi_i^* \widehat{f}) \right\|_{L_v^p} \right] \\ & = C_1 \cdot \sum_{i \in I} [B_{j,i} \cdot c_i] = C_1 \cdot \left(\vec{B} c \right)_j, \end{aligned} \quad (4.10)$$

where we defined $c_i := \left\| \mathcal{F}^{-1}(\varphi_i^* \widehat{f}) \right\|_{L_v^p}$ for all $i \in I$.

Setting $d_i := \|\mathcal{F}^{-1}(\varphi_i \hat{f})\|_{L_v^p}$ for $i \in I$ and using the triangle inequality for L_v^p , we get $c_i \leq (\Gamma_{\mathcal{Q}} d)_i$ for $i \in I$. By solidity of $\ell_w^q(I)$, this allows us to conclude

$$\begin{aligned} C_1 \cdot \|\vec{B}c\|_{\ell_w^q} &\leq C_1 \cdot \|\vec{B}\| \cdot \|c\|_{\ell_w^q} \\ &\leq C_1 \cdot \|\vec{B}\| \cdot \|\Gamma_{\mathcal{Q}} d\|_{\ell_w^q} \\ &\leq C_1 \cdot \|\Gamma_{\mathcal{Q}}\| \cdot \|\vec{B}\| \cdot \|d\|_{\ell_w^q} \\ &= C_1 \cdot \|\Gamma_{\mathcal{Q}}\| \cdot \|\vec{B}\| \cdot \|f\|_{\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)} < \infty. \end{aligned} \quad (4.11)$$

In particular, we get $(\vec{B}c)_j < \infty$ for all $j \in I$, so that the right-hand side of equation (4.10) is finite. We now use this estimate for $\delta = 1$: For arbitrary $x \in \mathbb{R}^d$ and $a \in T_j^{-T}[-1, 1]^d$, we have $x, x + a \in x + T_j^{-T}[-1, 1]^d$ and hence

$$\begin{aligned} &\left| \left[\mathcal{F}^{-1}(\widehat{\gamma^{(j)}} \cdot \varphi_i \cdot \hat{f}) \right](x + a) \right| \\ &\leq \left| \left(M_{-b_j} \left[\mathcal{F}^{-1}(\widehat{\gamma^{(j)}} \cdot \varphi_i \cdot \hat{f}) \right] \right)(x + a) - \left(M_{-b_j} \left[\mathcal{F}^{-1}(\widehat{\gamma^{(j)}} \cdot \varphi_i \cdot \hat{f}) \right] \right)(x) \right| + \left| \left(M_{-b_j} \left[\mathcal{F}^{-1}(\widehat{\gamma^{(j)}} \cdot \varphi_i \cdot \hat{f}) \right] \right)(x) \right| \\ &\leq \left[\text{osc}_{T_j^{-T}[-1, 1]^d} \left(M_{-b_j} \left[\mathcal{F}^{-1}(\widehat{\gamma^{(j)}} \cdot \varphi_i \cdot \hat{f}) \right] \right) \right](x) + \left| \mathcal{F}^{-1}(\widehat{\gamma^{(j)}} \cdot \varphi_i \cdot \hat{f})(x) \right|, \end{aligned}$$

which yields

$$M_{T_j^{-T}[-1, 1]^d} \left[\mathcal{F}^{-1}(\widehat{\gamma^{(j)}} \cdot \varphi_i \cdot \hat{f}) \right] \leq \text{osc}_{T_j^{-T}[-1, 1]^d} \left(M_{-b_j} \left[\mathcal{F}^{-1}(\widehat{\gamma^{(j)}} \cdot \varphi_i \cdot \hat{f}) \right] \right) + \left| \mathcal{F}^{-1}(\widehat{\gamma^{(j)}} \cdot \varphi_i \cdot \hat{f}) \right|.$$

Using the triangle inequality for $L_v^p(\mathbb{R}^d)$ and solidity of $L_v^p(\mathbb{R}^d)$, this yields

$$\begin{aligned} &\sum_{i \in I} \left\| \mathcal{F}^{-1}(\widehat{\gamma^{(j)}} \cdot \varphi_i \cdot \hat{f}) \right\|_{W_{T_j^{-T}[-1, 1]^d}(L_v^p)} \\ &= \sum_{i \in I} \left\| M_{T_j^{-T}[-1, 1]^d} \left[\mathcal{F}^{-1}(\widehat{\gamma^{(j)}} \cdot \varphi_i \cdot \hat{f}) \right] \right\|_{L_v^p} \\ &\leq \sum_{i \in I} \left(\left\| \mathcal{F}^{-1}(\widehat{\gamma^{(j)}} \cdot \varphi_i \cdot \hat{f}) \right\|_{L_v^p} + \left\| \text{osc}_{T_j^{-T}[-1, 1]^d} \left(M_{-b_j} \left[\mathcal{F}^{-1}(\widehat{\gamma^{(j)}} \cdot \varphi_i \cdot \hat{f}) \right] \right) \right\|_{L_v^p} \right) < \infty. \end{aligned}$$

Here, finiteness of the right-hand side follows from equation (4.10) (with $\delta = 1$) and from Theorem 3.5, where we saw that the series $\sum_{i \in I} \mathcal{F}^{-1}(\widehat{\gamma^{(j)}} \cdot \varphi_i \cdot \hat{f})$ converges normally in L_v^p .

But it follows from equation (2.13) that $W_{T_j^{-T}[-1, 1]^d}(L_v^p) \hookrightarrow L_{(1+|\bullet|)^{-K}}^{\infty}(\mathbb{R}^d)$, where the norm of the embedding might heavily depend on j . Setting $\|h\|_* := \sup_{x \in \mathbb{R}^d} (1 + |x|)^{-K} |h(x)|$, this allows us to conclude by continuity that

$$\sum_{i \in I} \left\| \mathcal{F}^{-1}(\widehat{\gamma^{(j)}} \cdot \varphi_i \cdot \hat{f}) \right\|_* = \sum_{i \in I} \left\| \mathcal{F}^{-1}(\widehat{\gamma^{(j)}} \cdot \varphi_i \cdot \hat{f}) \right\|_{L_{(1+|\bullet|)^{-K}}^{\infty}} \lesssim_j \sum_{i \in I} \left\| \mathcal{F}^{-1}(\widehat{\gamma^{(j)}} \cdot \varphi_i \cdot \hat{f}) \right\|_{W_{T_j^{-T}[-1, 1]^d}(L_v^p)} < \infty,$$

so that the series in equation (4.9) indeed converges *locally* uniformly. Hence, $\gamma^{(j)} * f$ is continuous for every $j \in I$ and arbitrary $f \in \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$.

Now, it is not hard to see $\text{osc}_{\mathcal{Q}}(\sum_{i \in I} f_i) \leq \sum_{i \in I} (\text{osc}_{\mathcal{Q}} f_i)$ for each pointwise convergent series $\sum_{i \in I} f_i$. Hence, equation (4.9) and the triangle inequality for $L_v^p(\mathbb{R}^d)$ imply

$$\begin{aligned} \frac{1}{\delta} \left\| \text{osc}_{\delta \cdot T_j^{-T}[-1, 1]^d} \left[M_{-b_j}(\gamma^{(j)} * f) \right] \right\|_{L_v^p} &\leq \frac{1}{\delta} \sum_{i \in I} \left\| \text{osc}_{\delta \cdot T_j^{-T}[-1, 1]^d} \left(M_{-b_j} \left[\mathcal{F}^{-1}(\widehat{\gamma^{(j)}} \cdot \varphi_i \cdot \hat{f}) \right] \right) \right\|_{L_v^p} \\ &\stackrel{(\text{eq. (4.10)})}{\leq} C_1 \cdot (\vec{B}c)_j < \infty \end{aligned}$$

for all $j \in I$ and $\delta \in (0, 1]$. By equation (4.11) and by solidity of $\ell_w^q(I)$, this yields

$$\left\| \left(\sup_{0 < \delta \leq 1} \frac{1}{\delta} \left\| \text{osc}_{\delta \cdot T_j^{-T}[-1, 1]^d} \left[M_{-b_j}(\gamma^{(j)} * f) \right] \right\|_{L_v^p} \right)_{j \in I} \right\|_{\ell_w^q} \leq C_1 \cdot \|\vec{B}c\|_{\ell_w^q} \leq C_1 \cdot \|\Gamma_{\mathcal{Q}}\| \cdot \|\vec{B}\| \cdot \|f\|_{\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)} < \infty.$$

Finally, Theorem 3.5 shows

$$\left\| \left(\left\| \gamma^{(j)} * f \right\|_{L_v^p} \right)_{j \in I} \right\|_{\ell_w^q} \leq \|\Gamma_{\mathcal{Q}}\| \cdot \|\vec{A}\| \cdot \|f\|_{\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)} < \infty.$$

It is not hard to see that this implies boundedness of Ana_{osc} , with a bound for the operator norm as in the statement of the lemma, since $2^{\max\{0, \frac{1}{q}-1\}}$ is a valid triangle constant for $\ell_w^q(I)$ and since $C_1 \geq 1$.

In case of $p \in (0, 1)$, we first note that equation (2.13) yields $V_j = W_{T_j^{-T}[-1,1]^d}(L_v^p) \hookrightarrow L_{(1+|\bullet|)^{-\kappa}}^\infty(\mathbb{R}^d)$, where again the norm of the embedding might depend heavily on the choice of $j \in I$. But as seen in Theorem 3.5, the series in equation (4.9) converges in V_j and hence in $L_{(1+|\bullet|)^{-\kappa}}^\infty(\mathbb{R}^d)$, which yields *locally* uniform convergence, since each summand of the series is continuous. In particular, we get continuity of $\gamma^{(j)} * f$ for each $j \in I$.

The remainder of the argument is similar as that for $p \in [1, \infty]$. Nevertheless, it needs to be modified slightly, since for $p \in (0, 1)$, $L_v^p(\mathbb{R}^d)$ does not satisfy the triangle inequality, but instead the so-called p -triangle inequality, i.e., $\|f + g\|_{L_v^p}^p \leq \|f\|_{L_v^p}^p + \|g\|_{L_v^p}^p$. Precisely, using equation (4.9) and the estimates $\text{osc}_Q(\sum_{i \in I} f_i) \leq \sum_{i \in I} (\text{osc}_Q f_i)$ and $M_Q(\sum_{i \in I} f_i) \leq \sum_{i \in I} (M_Q f_i)$, as well as the p -triangle inequality for $L_v^p(\mathbb{R}^d)$, we get for arbitrary $0 < \delta \leq 1$ that

$$\begin{aligned} & \left(\frac{1}{\delta} \left\| \text{osc}_{\delta \cdot T_j^{-T}[-1,1]^d} \left[M_{-b_j} \left(\gamma^{(j)} * f \right) \right] \right\|_{W_{T_j^{-T}[-1,1]^d}(L_v^p)} \right)^p \\ & \leq \frac{1}{\delta^p} \sum_{i \in I} \left\| \text{osc}_{\delta \cdot T_j^{-T}[-1,1]^d} \left(M_{-b_j} \left[\mathcal{F}^{-1} \left(\widehat{\gamma^{(j)}} \cdot \varphi_i \cdot \widehat{f} \right) \right] \right) \right\|_{W_{T_j^{-T}[-1,1]^d}(L_v^p)}^p \\ & \stackrel{(\text{Lemma 4.2})}{\leq} C_2^p \cdot \sum_{i \in I} \left[|\det T_i|^{1-p} \cdot (1 + \|T_j^{-1} T_i\|)^{pK+d} \cdot \|\mathcal{F}^{-1}(\varphi_i^* \widehat{f})\|_{L_v^p}^p \cdot \left\| \mathcal{F}^{-1} \left[\widehat{\phi^{(j)}} \cdot \varphi_i \right] \right\|_{L_{v_0}^p}^p \right] \\ & \left(\text{with } \theta_i := c_i^p = \|\mathcal{F}^{-1}(\varphi_i^* \widehat{f})\|_{L_v^p}^p \right) = C_2^p \cdot \sum_{i \in I} [B_{j,i} \theta_i] = C_2^p \cdot \left(\vec{B} \theta \right)_j, \end{aligned}$$

where the constant $C_2 > 0$ is provided by Lemma 4.2.

We conclude using the solidity of $\ell_w^q(I)$ that

$$\begin{aligned} & \left\| \left(\sup_{0 < \delta \leq 1} \frac{1}{\delta} \left\| \text{osc}_{\delta \cdot T_j^{-T}[-1,1]^d} \left[M_{-b_j} \left(\gamma^{(j)} * f \right) \right] \right\|_{W_{T_j^{-T}[-1,1]^d}(L_v^p)} \right) \right\|_{j \in I} \Big\|_{\ell_w^q} \\ & \leq C_2 \cdot \left\| \left(\vec{B} \theta \right)^{1/p} \right\|_{\ell_w^q} = C_2 \cdot \left\| \left(w^p \cdot \vec{B} \theta \right)^{1/p} \right\|_{\ell^q} \\ & = C_2 \cdot \left\| w^p \cdot \vec{B} \theta \right\|_{\ell^{q/p}}^{1/p} = C_2 \cdot \left\| \vec{B} \theta \right\|_{\ell_{w^{\min\{1,p\}}}^r}^{1/p} \\ & \leq C_2 \cdot \left(\|\vec{B}\| \cdot \|\theta\|_{\ell_{w^{\min\{1,p\}}}^r} \right)^{1/p} \\ & = C_2 \cdot \|\vec{B}\|^{1/p} \cdot \|w^p \cdot \theta\|_{\ell^{q/p}}^{1/p} \\ & = C_2 \cdot \|\vec{B}\|^{1/p} \cdot \|w \cdot \theta^{1/p}\|_{\ell^q} \\ & = C_2 \cdot \|\vec{B}\|^{1/p} \cdot \|c\|_{\ell_w^q}. \end{aligned}$$

Finally, using the quasi-triangle inequality $\left\| \sum_{i=1}^N f_i \right\|_{L^p} \leq N^{\frac{1}{p}-1} \cdot \sum_{i=1}^N \|f_i\|_{L^p}$ (cf. [41, Exercise 1.1.5(c)]) and the estimate $|i^*| \leq N_Q$ for all $i \in I$, we also get $c_i \leq N_Q^{\frac{1}{p}-1} \cdot (\Gamma_Q d)_i$ for all $i \in I$ and $d_i := \left\| \mathcal{F}^{-1}(\varphi_i \cdot \widehat{f}) \right\|_{L_v^p}$. Hence,

$$\left\| \left(\sup_{0 < \delta \leq 1} \frac{1}{\delta} \left\| \text{osc}_{\delta \cdot T_j^{-T}[-1,1]^d} \left[M_{-b_j} \left(\gamma^{(j)} * f \right) \right] \right\|_{W_{T_j^{-T}[-1,1]^d}(L_v^p)} \right) \right\|_{j \in I} \Big\|_{\ell_w^q} \leq C_2 \cdot N_Q^{\frac{1}{p}-1} \cdot \|\Gamma_Q\| \cdot \|\vec{B}\|^{1/p} \cdot \|d\|_{\ell_w^q}.$$

Because of $\|d\|_{\ell_w^q} = \|f\|_{\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)}$ and in combination with Theorem 3.5, we can now derive the claim using the same arguments as for $p \in [1, \infty]$. Here, we use that $C_Q \geq \|T_i^{-1} T_i\| = 1$, so that the constant $C_3 > 0$ provided by

Theorem 3.5 (for $p \in (0, 1)$) satisfies

$$\begin{aligned} C_3 &= N_{\mathcal{Q}}^{\frac{1}{p}-1} \cdot \left(12288 \cdot d^{3/2} \cdot \left\lceil K + \frac{d+1}{p} \right\rceil \right)^{\lceil K + \frac{d+1}{p} \rceil + 1} \cdot (1 + R_{\mathcal{Q}})^{d/p} (12R_{\mathcal{Q}}C_{\mathcal{Q}})^{d(\frac{1}{p}-1)} \cdot \Omega_0^K \Omega_1 \\ &\stackrel{(\text{since } \Omega_0, \Omega_1 \geq 1)}{\leq} N_{\mathcal{Q}}^{\frac{1}{p}-1} \cdot 12^{d(\frac{1}{p}-1)} \left(12288 \cdot d^{3/2} \cdot \left\lceil K + \frac{d+1}{p} \right\rceil \right)^{\lceil K + \frac{d+1}{p} \rceil + 1} \cdot (1 + C_{\mathcal{Q}}R_{\mathcal{Q}})^{d(\frac{2}{p}-1)} \cdot \Omega_0^{5K} \Omega_1^5, \\ &\leq N_{\mathcal{Q}}^{\frac{1}{p}-1} \cdot 12^{d(\frac{1}{p}-1)} \left(12288 \cdot d^{3/2} \cdot \left\lceil K + \frac{d+1}{p} \right\rceil \right)^{\lceil K + \frac{d+1}{p} \rceil + 1} \cdot (1 + C_{\mathcal{Q}}R_{\mathcal{Q}})^{\frac{2d}{p}} \cdot \Omega_0^{5K} \Omega_1^5, \end{aligned}$$

so that

$$\begin{aligned} \frac{C_3}{N_{\mathcal{Q}}^{\frac{1}{p}-1} C_2} &\leq 370 \cdot d^{11/2} \cdot d^{d/2p} \cdot \frac{12^{d(\frac{1}{p}-1)} \left(12288 \cdot d^{3/2} \cdot \left\lceil K + \frac{d+1}{p} \right\rceil \right)^{\lceil K + \frac{d+1}{p} \rceil + 1}}{2^{16\frac{d}{p}} \cdot \left(4032 \cdot d^3 \cdot \left\lceil K + \frac{d+1}{p} \right\rceil \right)^{2\lceil K + \frac{d+1}{p} \rceil + 2}} \\ &= \frac{370 \cdot d^{11/2} \cdot d^{d/2p} \cdot 12^{d(\frac{1}{p}-1)}}{2^{16\frac{d}{p}} \left(4032 \cdot d^3 \cdot \left\lceil K + \frac{d+1}{p} \right\rceil \right)^{\lceil K + \frac{d+1}{p} \rceil + 1}} \cdot \left(\frac{12288 \cdot d^{3/2}}{4032 \cdot d^3} \right)^{\lceil K + \frac{d+1}{p} \rceil + 1} \\ &\leq \frac{370 \cdot d^{11/2} \cdot d^{d/2p} \cdot 12^{\frac{d}{p}}}{2^{16\frac{d}{p}}} \cdot \left(\frac{4}{4032 \cdot d^{9/2}} \right)^{\lceil K + \frac{d+1}{p} \rceil + 1} \\ &\stackrel{(\text{since } \lceil K + \frac{d+1}{p} \rceil \geq \frac{d+1}{p} \geq \frac{d}{p} + 1)}{\leq} \frac{370 \cdot d^{11/2} \cdot d^{d/2p}}{2^{12\frac{d}{p}}} \cdot \left(\frac{1}{1000 \cdot d^{9/2}} \right)^{\frac{d}{p} + 2} \\ &\leq \frac{370}{1000000} \frac{d^{11/2} \cdot d^{d/2p}}{2^{12\frac{d}{p}}} \cdot d^{-\frac{9}{2}\frac{d}{p}} d^{-9} = \frac{370}{1000000} \frac{1}{2^{12\frac{d}{p}}} \cdot d^{-4\frac{d}{p}} d^{-7/2} \leq 1. \quad \square \end{aligned}$$

In view of the preceding lemma, we know that (if $\Gamma = (\gamma_i)_{i \in I}$ fulfills Assumption 4.1) each of the functions $\gamma^{(j)} * f$ and hence also $\gamma^{[j]} * f = |\det T_j|^{-1/2} \cdot \gamma^{(j)} * f$ is continuous, so that the coefficient mapping

$$\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q) \ni f \mapsto \left[\left(\gamma^{[j]} * f \right) (\delta \cdot T_j^{-T} k) \right]_{j \in I, k \in \mathbb{Z}^d} \in \mathbb{C}^{I \times \mathbb{Z}^d}$$

is well-defined. But eventually, we want to show that this map yields a Banach frame for $\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$, so that we also need to construct a suitable “reconstruction mapping”, which can recover f from these coefficients. The following lemma is an important ingredient for the construction of this reconstruction map.

Lemma 4.4. *For $i \in I$ and $0 < \delta \leq 1$, let*

$$\text{Synth}_{\delta,i} : \mathbb{C}^{\mathbb{Z}^d} \rightarrow \{f : \mathbb{R}^d \rightarrow \mathbb{C} \mid f \text{ measurable}\}, (c_k)_{k \in \mathbb{Z}^d} \mapsto M_{b_i} \left[\sum_{k \in \mathbb{Z}^d} c_k \cdot e^{-2\pi i \langle b_i, \delta \cdot T_i^{-T} k \rangle} \mathbf{1}_{\delta \cdot T_i^{-T}(k + [0,1]^d)} \right].$$

Then $\text{Synth}_{\delta,i}$ is well-defined and yields a bounded operator $\text{Synth}_{\delta,i} : C_i^{(\delta)} \rightarrow V_i$, where the i -th coefficient space $C_i^{(\delta)}$ is defined as in equation (4.2). More precisely, we have

$$\frac{\delta^{d/p}}{\Omega_0^K \Omega_1 \cdot (1 + \sqrt{d})^K} \cdot |\det T_i|^{-1/p} \cdot \|c\|_{C_i^{(\delta)}} \leq \|\text{Synth}_{\delta,i} c\|_{V_i} \leq C_{d,p,\delta,K} \cdot |\det T_i|^{-1/p} \cdot \|c\|_{C_i^{(\delta)}} \quad \forall c \in \mathbb{C}^{\mathbb{Z}^d},$$

with

$$C_{d,p,\delta,K} = \begin{cases} (1 + \sqrt{d})^K \cdot \Omega_0^K \Omega_1 \cdot \delta^{d/p}, & \text{if } p \in [1, \infty], \\ 4^{d/p} \cdot (1 + 2\sqrt{d})^K \cdot \Omega_0^K \Omega_1, & \text{if } p \in (0, 1). \end{cases} \quad \blacktriangleleft$$

Proof. First note that $\text{Synth}_{\delta,i}$ is well-defined, since the sets $(\delta \cdot T_i^{-T}(k + [0,1]^d))_{k \in \mathbb{Z}^d}$ are pairwise disjoint. Also, we can ignore the modulation M_{b_i} in the following, since $\|M_{b_i} f\|_{V_i} = \|f\|_{V_i}$, because of $\|f\|_{V_i} = \|g\|_{V_i}$ for measurable

f, g satisfying $|f| = |g|$. Furthermore, since we have for $x \in \delta \cdot T_i^{-T} \left(k + [0, 1]^d \right)$, i.e., for $x = \delta \cdot T_i^{-T} k + \delta T_i^{-T} q$ with $q \in [0, 1]^d$ that

$$\begin{aligned} v_k^{(i, \delta)} &= v(\delta \cdot T_i^{-T} k) = v(x - \delta T_i^{-T} q) \leq v(x) \cdot v_0(-\delta T_i^{-T} q) \\ (\text{assump. on } v_0) &\leq \Omega_1 \cdot (1 + |\delta T_i^{-T} q|)^K \cdot v(x) \\ (\text{eq. (1.11)}) &\leq \Omega_0^K \Omega_1 \cdot (1 + |\delta q|)^K \cdot v(x) \\ (\text{since } \delta \leq 1) &\leq \Omega_0^K \Omega_1 \cdot (1 + \sqrt{d})^K \cdot v(x), \end{aligned}$$

Lemma 2.2 implies

$$\begin{aligned} \|\text{Synth}_{\delta, i}(c_k)_{k \in \mathbb{Z}^d}\|_{V_i} &\geq \|\text{Synth}_{\delta, i}(c_k)_{k \in \mathbb{Z}^d}\|_{L_v^p} \\ (\text{pairwise disjointness}) &= \left[\sum_{k \in \mathbb{Z}^d} |c_k|^p \int_{\delta \cdot T_i^{-T}(k + [0, 1]^d)} [v(x)]^p dx \right]^{1/p} \\ &\geq \frac{1}{\Omega_0^K \Omega_1 \cdot (1 + \sqrt{d})^K} \cdot \left[\sum_{k \in \mathbb{Z}^d} \left| v_k^{(i, \delta)} \cdot c_k \right|^p \cdot \lambda_d(\delta \cdot T_i^{-T}[k + [0, 1]^d]) \right]^{1/p} \\ &= \frac{\delta^{d/p} \cdot |\det T_i|^{-1/p}}{\Omega_0^K \Omega_1 \cdot (1 + \sqrt{d})^K} \cdot \|(c_k)_{k \in \mathbb{Z}^d}\|_{C_i^{(\delta)}}. \end{aligned} \quad (4.12)$$

This proves the lower bound.

Now, we establish the reverse inequality for $p \in [1, \infty]$: For $x = \delta \cdot T_i^{-T} k + \delta T_i^{-T} q \in \delta \cdot T_i^{-T} \left(k + [-L, L]^d \right)$ with $L \geq 1$, we have as above that

$$\begin{aligned} v(x) &= v(\delta \cdot T_i^{-T} k + \delta \cdot T_i^{-T} q) \\ &\leq v(\delta \cdot T_i^{-T} k) \cdot v_0(\delta \cdot T_i^{-T} q) \\ (\text{assump. on } v_0 \text{ and eq. (1.11)}) &\leq \Omega_0^K \Omega_1 \cdot v_k^{(i, \delta)} \cdot (1 + |\delta \cdot q|)^K \\ (\text{since } \delta \leq 1) &\leq (1 + L\sqrt{d})^K \cdot \Omega_0^K \Omega_1 \cdot v_k^{(i, \delta)}. \end{aligned} \quad (4.13)$$

Furthermore, since $p \in [1, \infty]$, the first inequality in estimate (4.12) from above is actually an equality, so that

$$\begin{aligned} \|\text{Synth}_{\delta, i}(c_k)_{k \in \mathbb{Z}^d}\|_{V_i} &= \left[\sum_{k \in \mathbb{Z}^d} |c_k|^p \int_{\delta \cdot T_i^{-T}(k + [0, 1]^d)} [v(x)]^p dx \right]^{1/p} \\ &\leq (1 + \sqrt{d})^K \cdot \Omega_0^K \Omega_1 \cdot \left[\sum_{k \in \mathbb{Z}^d} \left| v_k^{(i, \delta)} c_k \right|^p \cdot \lambda_d(\delta \cdot T_i^{-T}[k + [0, 1]^d]) \right]^{1/p} \\ &= (1 + \sqrt{d})^K \cdot \Omega_0^K \Omega_1 \cdot \delta^{d/p} \cdot |\det T_i|^{-1/p} \cdot \|(c_k)_{k \in \mathbb{Z}^d}\|_{C_i^{(\delta)}}. \end{aligned}$$

Finally, for $p \in (0, 1)$, we use the estimate $M_Q(\sum_{i \in I} f_i) \leq \sum_{i \in I} M_Q f_i$ and the p -triangle inequality for $L_v^p(\mathbb{R}^d)$ to deduce

$$\begin{aligned} \|\text{Synth}_{\delta, i}(c_k)_{k \in \mathbb{Z}^d}\|_{V_i}^p &\leq \left\| \sum_{k \in \mathbb{Z}^d} M_{T_i^{-T}[-1, 1]^d} \left(c_k \cdot e^{-2\pi i \langle b_i, \delta \cdot T_i^{-T} k \rangle} \mathbb{1}_{\delta \cdot T_i^{-T}(k + [0, 1]^d)} \right) \right\|_{L_v^p}^p \\ &\leq \sum_{k \in \mathbb{Z}^d} \left[|c_k|^p \cdot \left\| M_{T_i^{-T}[-1, 1]^d} \mathbb{1}_{\delta \cdot T_i^{-T}(k + [0, 1]^d)} \right\|_{L_v^p}^p \right]. \end{aligned}$$

Next, observe

$$\left(M_{T_i^{-T}[-1, 1]^d} \mathbb{1}_{\delta \cdot T_i^{-T}(k + [0, 1]^d)} \right)(x) = \left\| \mathbb{1}_{\delta \cdot T_i^{-T}(k + [0, 1]^d)} \cdot \mathbb{1}_{x + T_i^{-T}[-1, 1]^d} \right\|_{L^\infty} \leq 1.$$

Furthermore, if the function inside the $\|\bullet\|_{L^\infty}$ norm does not vanish identically, we have

$$x \in \delta T_i^{-T} k + T_i^{-T} \left(\delta [0, 1]^d - [-1, 1]^d \right) \subset \delta T_i^{-T} k + T_i^{-T} [-2, 2]^d.$$

Hence, equation (4.13) yields $v(x) \leq \left(1 + 2\sqrt{d}\right)^K \Omega_0^K \Omega_1 \cdot v_k^{(i, \delta)}$ and thus

$$\begin{aligned} \left\| M_{T_i^{-T}[-1, 1]^d} \mathbb{1}_{\delta \cdot T_i^{-T}(k + [0, 1]^d)} \right\|_{L_v^p}^p &\leq \left(\left(1 + 2\sqrt{d}\right)^K \Omega_0^K \Omega_1 \right)^p \cdot \left[v_k^{(i, \delta)} \right]^p \cdot \lambda_d \left(\delta T_i^{-T} k + T_i^{-T} [-2, 2]^d \right) \\ &= 4^d \cdot \left(\left(1 + 2\sqrt{d}\right)^K \Omega_0^K \Omega_1 \right)^p \cdot |\det T_i|^{-1} \cdot \left[v_k^{(i, \delta)} \right]^p, \end{aligned}$$

so that we get

$$\left\| \text{Synth}_{\delta, i}(c_k)_{k \in \mathbb{Z}^d} \right\|_{V_i}^p \leq 4^d \cdot \left(\left(1 + 2\sqrt{d}\right)^K \Omega_0^K \Omega_1 \right)^p \cdot |\det T_i|^{-1} \cdot \left\| (c_k)_{k \in \mathbb{Z}^d} \right\|_{C_i^{(\delta)}}^p,$$

as claimed. \square

Below, we will employ a Neumann series argument to construct the reconstruction operator R . To this end, we need to know that the space $\ell_w^q([V_i]_{i \in I})$ is a Quasi-Banach space.

Lemma 4.5. *Let $(X_i)_{i \in I}$ be a sequence of Quasi-Banach spaces, each with $\|x + y\|_{X_i} \leq C_i \cdot (\|x\|_{X_i} + \|y\|_{X_i})$ for all $x, y \in X_i$ and suitable $C_i \geq 1$. Assume that $C := \sup_{i \in I} C_i$ is finite and that each quasi-norm $\|\bullet\|_{X_i}$ is continuous.*

Define

$$\ell_w^q([X_i]_{i \in I}) := \left\{ x = (x_i)_{i \in I} \in \prod_{i \in I} X_i \mid \|x\| := \left\| (\|x_i\|_{X_i})_{i \in I} \right\|_{\ell_w^q(I)} < \infty \right\}.$$

Then $(\ell_w^q([X_i]_{i \in I}), \|\bullet\|)$ is a Quasi-Banach space. \blacktriangleleft

Remark. The lemma applies in particular with the choice $X_i = V_i$. Indeed, $\|\bullet\|_{L_v^p}$ is an s -norm for $s := \min\{1, p\}$; since $M_Q(f + g) \leq M_Q f + M_Q g$, we get $\|f + g\|_{W_Q(L_v^p)}^s \leq \|f\|_{W_Q(L_v^p)}^s + \|g\|_{W_Q(L_v^p)}^s$, so that $\|\bullet\|_{W_Q(L_v^p)}$ is also an s -norm and hence continuous, since $\|x_n\|^s - \|x\|^s \leq \|x_n - x\|^s \xrightarrow{n \rightarrow \infty} 0$ for any s -norm $\|\bullet\|$ if $\|x_n - x\| \xrightarrow{n \rightarrow \infty} 0$. Furthermore, in case of $p \in [1, \infty]$, one can choose $C_i = 1$ for all $i \in I$. Finally, for $p \in (0, 1)$, Remark 3.2 shows that each V_i is a Quasi-Banach space and that we can choose $C_i = 2^{\frac{1}{p}-1}$ for all $i \in I$. Hence, $V = \ell_w^q([V_i]_{i \in I})$ is a Quasi-Banach space. \blacktriangleright

Proof. For brevity, let $X := \ell_w^q([X_i]_{i \in I})$. It is clear that X is closed under multiplication with scalars and that $\|\alpha \cdot x\| = |\alpha| \cdot \|x\|$ for $\alpha \in \mathbb{K}$ (with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$) and $x \in X$. Furthermore, if $\|x\| = 0$ for $x = (x_i)_{i \in I}$, then $\|x_i\|_{X_i} = 0$ for all $i \in I$, so that $x = 0$. Finally, for $x, y \in X$, we have by solidity of $\ell_w^q(I)$ that

$$\begin{aligned} \|x + y\| &= \left\| (\|x_i + y_i\|_{X_i})_{i \in I} \right\|_{\ell_w^q} \leq \left\| (C \cdot [\|x_i\|_{X_i} + \|y_i\|_{X_i}])_{i \in I} \right\|_{\ell_w^q} \\ &\leq C \cdot C_q \cdot \left[\left\| (\|x_i\|_{X_i})_{i \in I} \right\|_{\ell_w^q} + \left\| (\|y_i\|_{X_i})_{i \in I} \right\|_{\ell_w^q} \right] \\ &= C \cdot C_q \cdot [\|x\| + \|y\|] < \infty, \end{aligned}$$

where C_q is a triangle constant for $\ell_w^q(I)$. Hence, X is closed under addition (and thus a vector space as a subspace of $\prod_{i \in I} X_i$) and $\|\bullet\|$ is a quasi-norm on X .

Now, let $(x^{(n)})_{n \in \mathbb{N}} = [(x_i^{(n)})_{i \in I}]_{n \in \mathbb{N}}$ be a Cauchy sequence in X . It is not hard to see that each of the projections $\pi_i : X \rightarrow X_i, (x_j)_{j \in I} \mapsto x_i$ is a bounded linear map, so that each sequence $(x_i^{(n)})_{n \in \mathbb{N}}$ is Cauchy in X_i and hence convergent to some $x_i \in X_i$. Now, let $\varepsilon > 0$ be arbitrary. There is some $N_0 \in \mathbb{N}$ satisfying $\|x^{(n)} - x^{(m)}\| \leq \varepsilon$ for all $n, m \geq N_0$. By Fatou's lemma and by continuity of $\|\bullet\|_{X_i}$, this implies for $m \geq N_0$ that

$$\begin{aligned} \left\| (\|x_i - x_i^{(m)}\|_{X_i})_{i \in I} \right\|_{\ell_w^q} &= \left\| \left(\liminf_{n \rightarrow \infty} \|x_i^{(n)} - x_i^{(m)}\|_{X_i} \right)_{i \in I} \right\|_{\ell_w^q} \leq \liminf_{n \rightarrow \infty} \left\| (\|x_i^{(n)} - x_i^{(m)}\|_{X_i})_{i \in I} \right\|_{\ell_w^q} \\ &= \liminf_{n \rightarrow \infty} \|x^{(n)} - x^{(m)}\| \leq \varepsilon < \infty. \end{aligned}$$

Since X is a vector space, this implies $x = (x_i)_{i \in I} = (x - x^{(m)}) + x^{(m)} \in X$, as well as $\|x - x^{(m)}\| \xrightarrow{m \rightarrow \infty} 0$. \square

The next lemma is our final preparation for proving that the coefficient map

$$f \mapsto \left[\left(\gamma^{[j]} * f \right) (\delta \cdot T_j^{-T} k) \right]_{j \in I, k \in \mathbb{Z}^d}$$

indeed yields a Banach frame for $\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$. This lemma essentially yields a replacement for the usual reproducing kernel property which is used in the theory of coorbit spaces (cf. [25, 26, 27, 67] and [76, Section 2]).

Lemma 4.6. *Assume that $\Gamma = (\gamma_i)_{i \in I}$ satisfies Assumptions 4.1 and 3.6. We clearly have a norm-decreasing embedding $W_j \hookrightarrow V_j$ and hence also $\iota : \ell_w^q([W_j]_{j \in I}) \hookrightarrow \ell_w^q([V_j]_{j \in I})$.*

Let

$$\text{Ana}_{\text{osc}} : \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q) \rightarrow \ell_w^q([W_j]_{j \in I}), f \mapsto (\gamma^{(j)} * f)_{j \in I}$$

as in Lemma 4.3, let

$$\text{Synth}_{\mathcal{D}} : \ell_w^q([V_j]_{j \in I}) \rightarrow \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q), (f_i)_{i \in I} \mapsto \sum_{i \in I} [\mathcal{F}^{-1}(\varphi_i \cdot \widehat{f}_i)]$$

be defined as in Lemma 3.9 and let

$$m_{\theta} : \ell_w^q([V_j]_{j \in I}) \rightarrow \ell_w^q([V_j]_{j \in I}), (f_j)_{j \in I} \mapsto [(\mathcal{F}^{-1} \theta_j) * f_j]_{j \in I}$$

be defined as in Lemma 3.8.

Then, the map

$$F : \ell_w^q([V_j]_{j \in I}) \rightarrow \ell_w^q([V_j]_{j \in I}), \quad F := \iota \circ \text{Ana}_{\text{osc}} \circ \text{Synth}_{\mathcal{D}} \circ m_{\theta}$$

is well-defined and bounded and satisfies the following additional properties:

- (1) $F \left[(\gamma^{(j)} * f)_{j \in I} \right] = (\gamma^{(j)} * f)_{j \in I}$ for all $f \in \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$.
- (2) $\text{Synth}_{\mathcal{D}} \circ m_{\theta} \circ \iota \circ \text{Ana}_{\text{osc}} = \text{id}_{\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)}$.
- (3) $F \circ F = F$.
- (4) The space $\Upsilon := \{(f_i)_{i \in I} \in \ell_w^q([V_i]_{i \in I}) \mid F(f_i)_{i \in I} = (f_i)_{i \in I}\}$ is a closed subspace of $\ell_w^q([V_i]_{i \in I})$.
- (5) For each $f = (f_i)_{i \in I} \in \Upsilon$, we have that each $f_i : \mathbb{R}^d \rightarrow \mathbb{C}$ is continuous and furthermore

$$\left\| \left[\text{osc}_{\delta \cdot T_i^{-T}[-1,1]^d} (M_{-b_i} f_i) \right]_{i \in I} \right\|_{\ell_w^q([V_i]_{i \in I})} \leq \|F_0\| \cdot \delta \cdot \|f\|_{\ell_w^q([V_i]_{i \in I})} \quad \forall \delta \in (0, 1] \quad (4.14)$$

for $F_0 := \text{Ana}_{\text{osc}} \circ \text{Synth}_{\mathcal{D}} \circ m_{\theta} : \ell_w^q([V_i]_{i \in I}) \rightarrow \ell_w^q([W_i]_{i \in I})$. Here, we have

$$\|F_0\| \leq 2^{\frac{1}{d}} C_{\mathcal{Q}, \Phi, v_0, p}^2 \cdot \|\Gamma_{\mathcal{Q}}\|^2 \cdot \left(\|\vec{A}\|^{\max\{1, \frac{1}{p}\}} + \|\vec{B}\|^{\max\{1, \frac{1}{p}\}} \right) \cdot C,$$

for $N := \left\lceil K + \frac{d+1}{\min\{1, p\}} \right\rceil$ and

$$C := \begin{cases} \frac{(2^{16} \cdot 768 / d^{\frac{3}{2}})^{\frac{d}{p}}}{2^{42} \cdot 12^d \cdot d^{15}} \cdot \left(2^{52} \cdot d^{\frac{25}{2}} \cdot N^3 \right)^{N+1} \cdot N_{\mathcal{Q}}^{2(\frac{1}{p}-1)} (1 + R_{\mathcal{Q}} C_{\mathcal{Q}})^{d(\frac{4}{p}-1)} \cdot \Omega_0^{13K} \Omega_1^{13} \Omega_2^{(p,K)}, & \text{if } p < 1, \\ \frac{1}{\sqrt{d} \cdot 2^{12+6\lceil K \rceil}} \cdot (2^{17} \cdot d^{5/2} \cdot N)^{\lceil K \rceil + d + 2} \cdot (1 + R_{\mathcal{Q}})^d \cdot \Omega_0^{3K} \Omega_1^3 \Omega_2^{(p,K)}, & \text{if } p \geq 1. \end{cases} \quad \blacktriangleleft$$

Proof. As a consequence of Lemmas 3.9, 3.8 and 4.3, we see that $F_0 : \ell_w^q([V_i]_{i \in I}) \rightarrow \ell_w^q([W_i]_{i \in I})$ is bounded with $\|F_0\| \leq \|\text{Ana}_{\text{osc}}\| \cdot \|\text{Synth}_{\mathcal{D}}\| \cdot \|m_{\theta}\|$. By plugging in the estimates for the norms of these operators which were obtained in the respective lemmas and using elementary estimates, we easily get the stated estimate for $\|F_0\|$. With F_0 , also $F = \iota \circ F_0$ is bounded. We now verify the different claims individually.

- (1) The assumptions of the current lemma include those of Theorem 3.10, where it was shown (cf. equation (3.9)) that $\text{id}_{\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)} = \text{Synth}_{\mathcal{D}} \circ m_{\theta} \circ \text{Ana}_{\Gamma}$, where $\text{Ana}_{\Gamma} = \iota \circ \text{Ana}_{\text{osc}}$. Hence,

$$\text{Synth}_{\mathcal{D}} \circ m_{\theta} \circ \iota \circ \text{Ana}_{\text{osc}} = \text{id}_{\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)}, \quad (4.15)$$

which proves the second part of the current lemma.

Furthermore, for $f \in \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$, we have $(\gamma^{(j)} * f)_{j \in I} = \text{Ana}_{\text{osc}} f \in \ell_w^q([W_i]_{i \in I}) \subset \ell_w^q([V_i]_{i \in I})$, so that $F\left([\gamma^{(j)} * f]_{j \in I}\right) \in \ell_w^q([V_i]_{i \in I})$ is well-defined. Finally, we get

$$\begin{aligned} F\left([\gamma^{(j)} * f]_{j \in I}\right) &= (\iota \circ \text{Ana}_{\text{osc}}) \left[(\text{Synth}_{\mathcal{D}} \circ m_{\theta}) [\gamma^{(j)} * f]_{j \in I} \right] \\ &= (\iota \circ \text{Ana}_{\text{osc}}) [(\text{Synth}_{\mathcal{D}} \circ m_{\theta} \circ \iota \circ \text{Ana}_{\text{osc}}) f] \\ &\stackrel{(\text{eq. (4.15)})}{=} \iota(\text{Ana}_{\text{osc}} f) = \iota \left[\left(\gamma^{(j)} * f \right)_{j \in I} \right] = \left(\gamma^{(j)} * f \right)_{j \in I}, \end{aligned}$$

as claimed in the first part.

(2) This was proved just above.

(3) As a consequence of equation (4.15) (i.e., of the second part of the lemma), we get

$$F \circ F = \iota \circ \text{Ana}_{\text{osc}} \circ \underbrace{\text{Synth}_{\mathcal{D}} \circ m_{\theta} \circ \iota \circ \text{Ana}_{\text{osc}}}_{=\text{id}_{\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)}} \circ \text{Synth}_{\mathcal{D}} \circ m_{\theta} = \iota \circ \text{Ana}_{\text{osc}} \circ \text{Synth}_{\mathcal{D}} \circ m_{\theta} = F.$$

(4) This trivially follows from continuity and linearity of F .

(5) For $(f_i)_{i \in I} \in \Upsilon$, we have $(f_i)_{i \in I} = F(f_i)_{i \in I} = \iota \circ F_0(f_i)_{i \in I}$ and hence $(f_i)_{i \in I} = F_0(f_i)_{i \in I}$, where—strictly speaking—on the left-hand side, $(f_i)_{i \in I}$ is interpreted as an element of $\ell_w^q([W_j]_{j \in I})$ and on the right-hand side as an element of $\ell_w^q([V_i]_{i \in I})$. In particular, since $W_j \leq C(\mathbb{R}^d)$, we see that each $f_i : \mathbb{R}^d \rightarrow \mathbb{C}$ is continuous. Finally, using boundedness of F_0 , we get

$$\begin{aligned} \sup_{0 < \delta \leq 1} \frac{1}{\delta} \left\| \left(\text{osc}_{\delta \cdot T_i^{-T}[-1,1]^d} [M_{-b_i} f_i] \right)_{i \in I} \right\|_{\ell_w^q([V_i]_{i \in I})} &\leq \|(f_i)_{i \in I}\|_{\ell_w^q([W_i]_{i \in I})} \\ &= \|F_0(f_i)_{i \in I}\|_{\ell_w^q([W_i]_{i \in I})} \\ &\leq \|F_0\| \cdot \|(f_i)_{i \in I}\|_{\ell_w^q([V_i]_{i \in I})}, \end{aligned}$$

which easily yields the claim. \square

Given all of these preparations, we can finally show that we obtain Banach frames for $\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$ in the expected way:

Theorem 4.7. Assume that $\Gamma = (\gamma_i)_{i \in I}$ satisfies Assumptions 4.1 and 3.6. Then there is some $\delta_0 > 0$ such that for every $0 < \delta \leq \delta_0$, the family $\left(L_{\delta \cdot T_i^{-T} k} \gamma^{[i]} \right)_{i \in I, k \in \mathbb{Z}^d}$ forms a Banach frame for $\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$, with $\widetilde{\gamma^{[i]}}(x) = \gamma^{[i]}(-x)$ and

$$\gamma^{[i]} = |\det T_i|^{1/2} \cdot M_{b_i} [\gamma_i \circ T_i^T] \quad \forall i \in I.$$

In fact, one can choose $\delta_0 = \frac{1}{1+2\|F_0\|^2}$, with F_0 as in Lemma 4.6.

Precisely, the Banach frame property has to be understood as follows:

- The **analysis operator**

$$A_{\delta} : \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q) \rightarrow \ell^q \left(|\det T_i|^{\frac{1}{2} - \frac{1}{p} \cdot w_i} \right)_{i \in I} \left([C_i^{(\delta)}]_{i \in I} \right), f \mapsto \left([\gamma^{[i]} * f] (\delta \cdot T_i^{-T} k) \right)_{k \in \mathbb{Z}^d, i \in I}$$

is well-defined and bounded for each $\delta \in (0, 1]$.

- As long as $0 < \delta \leq \delta_0$, there is a bounded linear **reconstruction operator**

$$R_{\delta} : \ell^q \left(|\det T_i|^{\frac{1}{2} - \frac{1}{p} \cdot w_i} \right)_{i \in I} \left([C_i^{(\delta)}]_{i \in I} \right) \rightarrow \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$$

satisfying $R_{\delta} \circ A_{\delta} = \text{id}_{\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)}$.

Finally, we also have the following **consistency property**: If $p_1, p_2, q_1, q_2 \in (0, \infty]$, if $w^{(1)} = (w_i^{(1)})_{i \in I}$ and $w^{(2)} = (w_i^{(2)})_{i \in I}$ are \mathcal{Q} -moderate weights and if $v^{(1)}, v^{(2)} : \mathbb{R}^d \rightarrow \mathbb{C}$ are weights such that the assumptions of the

current theorem are satisfied for $\mathcal{D}(\mathcal{Q}, L_{v^{(i)}}^{p_i}, \ell_{w^{(i)}}^{q_i})$ for $i \in \{1, 2\}$ and if $0 < \delta \leq \min\{\delta_1, \delta_2\}$, where the constant δ_i is equal to the constant δ_0 for the choices $p = p_i, q = q_i, w = w^{(i)}$ and $v = v^{(i)}$, then we have

$$\forall f \in \mathcal{D}(\mathcal{Q}, L_{v^{(2)}}^{p_2}, \ell_{w^{(2)}}^{q_2}) : f \in \mathcal{D}(\mathcal{Q}, L_{v^{(1)}}^{p_1}, \ell_{w^{(1)}}^{q_1}) \iff \left[(\gamma^{[j]} * f)(\delta \cdot T_j^{-T} k) \right]_{k \in \mathbb{Z}^d, j \in I} \in \ell_{\left(|\det T_j|^{\frac{1}{2} - \frac{1}{p_1}} w_j^{(1)} \right)_{j \in I}}^{q_1} \left(\left[C_j^{(1, \delta)} \right]_{j \in I} \right),$$

with $C_j^{(1, \delta)} = \ell_{(v^{(1)})^{(j, \delta)}}^p(\mathbb{Z}^d)$ and $(v^{(1)})_k^{(j, \delta)} = v^{(1)}(\delta \cdot T_j^{-T} k)$ for $j \in I$ and $k \in \mathbb{Z}^d$. \blacktriangleleft

Remark. • The statement of the theorem that the family $\left(L_{\delta \cdot T_i^{-T} k} \widetilde{\gamma^{[i]}} \right)_{i \in I, k \in \mathbb{Z}^d}$ forms a Banach frame for the decomposition space $\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$ has to be taken with a grain of salt (i.e., as saying that A_δ, R_δ as in the statement of the theorem are bounded and $R_\delta \circ A_\delta = \text{id}_{\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)}$). But if we have $\mathcal{O} = \mathbb{R}^d, \gamma_i \in \mathcal{S}(\mathbb{R}^d)$ for all $i \in I$ and $\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$, then this statement can be taken literally: As seen in the remark after Theorem 3.5, the definition of $\gamma^{[i]} * f$ given there coincides with the usual interpretation for $f \in \mathcal{S}'(\mathbb{R}^d) \supset \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$, so that we indeed have

$$A_\delta f = \left((\gamma^{[i]} * f)(\delta \cdot T_i^{-T} k) \right)_{k \in \mathbb{Z}^d, i \in I} = \left(\left\langle f, L_{\delta \cdot T_i^{-T} k} \widetilde{\gamma^{[i]}} \right\rangle_{\mathcal{S}', \mathcal{S}} \right)_{k \in \mathbb{Z}^d, i \in I}.$$

- For the consistency statement, note that we only claim that an equivalence of the form

$$f \in \mathcal{D}(\mathcal{Q}, L_{v^{(1)}}^{p_1}, \ell_{w^{(1)}}^{q_1}) \iff \left[(\gamma^{[j]} * f)(\delta \cdot T_j^{-T} k) \right]_{k \in \mathbb{Z}^d, j \in I} \in \ell_{\left(|\det T_j|^{\frac{1}{2} - \frac{1}{p_1}} w_j^{(1)} \right)_{j \in I}}^{q_1} \left(\left[C_j^{(1, \delta)} \right]_{j \in I} \right)$$

holds under the *assumption* that we *already know* $f \in \mathcal{D}(\mathcal{Q}, L_{v^{(2)}}^{p_2}, \ell_{w^{(2)}}^{q_2})$ for suitable $p_2, q_2, v^{(2)}, w^{(2)}$. In other words, we require that we already know that f has a certain *minimal amount of regularity*. This is quite natural, since for an arbitrary $f \in \mathcal{Z}'(\mathcal{O})$, there is no reason why $\gamma^{[j]} * f$ should be defined at all.

- As the proof will show, the action of R_δ on a given sequence $(c_k^{(i)})_{i \in I, k \in \mathbb{Z}^d} \in \ell_{\left(|\det T_i|^{\frac{1}{2} - \frac{1}{p}} w_i \right)_{i \in I}}^q \left([C_i^{(\delta)}]_{i \in I} \right)$ is actually *independent* of p, q, v, w . The only thing which depends on these quantities is δ_0 , so that $R_\delta(c_k^{(i)})_{i \in I, k \in \mathbb{Z}^d}$ is only defined for $0 < \delta \leq \delta_0 = \delta_0(p, q, v, w, \gamma)$. But once this is satisfied, the definition is independent of p, q, v, w . \blacklozenge

Proof. First of all, we remark that the L^2 -normalized functions $\gamma^{[i]}$ yield a nice statement of the theorem, while the proof can be formulated easier in terms of the L^1 -normalized functions $\gamma^{(i)}$. Hence, we introduce the isometric isomorphism

$$J : \ell_{\left(|\det T_i|^{\frac{1}{2} - \frac{1}{p}} w_i \right)_{i \in I}}^q \left([C_i^{(\delta)}]_{i \in I} \right) \rightarrow \ell_{\left(|\det T_i|^{-1/p} w_i \right)_{i \in I}}^q \left([C_i^{(\delta)}]_{i \in I} \right), (c_k^{(i)})_{k \in \mathbb{Z}^d, i \in I} \mapsto \left(|\det T_i|^{1/2} \cdot c_k^{(i)} \right)_{k \in \mathbb{Z}^d, i \in I}.$$

Then, we define $A_\delta^{(0)} := J \circ A_\delta$ and note

$$A_\delta^{(0)} f = \left((\gamma^{(i)} * f)(\delta \cdot T_i^{-T} k) \right)_{k \in \mathbb{Z}^d, i \in I} \quad \forall f \in \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q),$$

so it suffices to show that $A_\delta^{(0)} : \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q) \rightarrow \ell_{\left(|\det T_i|^{-1/p} w_i \right)_{i \in I}}^q \left([C_i^{(\delta)}]_{i \in I} \right)$ is well-defined and bounded. Further, if there is a bounded operator $R_\delta^{(0)} : \ell_{\left(|\det T_i|^{-1/p} w_i \right)_{i \in I}}^q \left([C_i^{(\delta)}]_{i \in I} \right) \rightarrow \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$ satisfying $R_\delta^{(0)} \circ A_\delta^{(0)} = \text{id}_{\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)}$, then a suitable definition of the reconstruction operator R_δ in the statement of the theorem is given by $R_\delta := R_\delta^{(0)} \circ J$, because of $R_\delta \circ A_\delta = R_\delta^{(0)} \circ J \circ J^{-1} \circ A_\delta^{(0)} = R_\delta^{(0)} \circ A_\delta^{(0)}$.

These considerations also apply to the consistency statement at the end of the theorem. All in all, we can thus replace $\gamma^{[i]}$ by $\gamma^{(i)}$ in the proof, as long as we replace all occurrences of $\ell_{\left(|\det T_i|^{\frac{1}{2} - \frac{1}{p}} w_i \right)_{i \in I}}^q \left([C_i^{(\delta)}]_{i \in I} \right)$ by $\ell_{\left(|\det T_i|^{-1/p} w_i \right)_{i \in I}}^q \left([C_i^{(\delta)}]_{i \in I} \right)$.

In the whole proof, we will use the nomenclature introduced in Lemma 4.6. As noted in that lemma, every function f_i is continuous if $(f_i)_{i \in I} \in \Upsilon$. Hence, for each $i \in I$, the operator

$$\text{Samp}_{\delta, i} : \Upsilon \rightarrow \mathbb{C}^{\mathbb{Z}^d}, (f_j)_{j \in I} \mapsto [f_i(\delta \cdot T_i^{-T} k)]_{k \in \mathbb{Z}^d}$$

is well-defined. Now, note with $\text{Synth}_{\delta,i}$ as in Lemma 4.4 that

$$\begin{aligned}
& \left| f_i(x) - \left[(\text{Synth}_{\delta,i} \circ \text{Samp}_{\delta,i})(f_j)_{j \in I} \right](x) \right| \\
&= \left| f_i(x) - \left(M_{b_i} \left[\sum_{k \in \mathbb{Z}^d} f_i(\delta \cdot T_i^{-T} k) \cdot e^{-2\pi i \langle b_i, \delta \cdot T_i^{-T} k \rangle} \mathbf{1}_{\delta \cdot T_i^{-T}(k+[0,1]^d)} \right] \right)(x) \right| \\
&= \left| (M_{-b_i} f_i)(x) - \sum_{k \in \mathbb{Z}^d} (M_{-b_i} f_i)(\delta \cdot T_i^{-T} k) \cdot \mathbf{1}_{\delta \cdot T_i^{-T}(k+[0,1]^d)}(x) \right| \\
& \left(\mathbb{R}^d = \biguplus_{k \in \mathbb{Z}^d} \delta T_i^{-T}(k+[0,1]^d) \right) = \left| \sum_{k \in \mathbb{Z}^d} \mathbf{1}_{\delta \cdot T_i^{-T}(k+[0,1]^d)}(x) \cdot [(M_{-b_i} f_i)(x) - (M_{-b_i} f_i)(\delta \cdot T_i^{-T} k)] \right| \\
&\leq \sum_{k \in \mathbb{Z}^d} \mathbf{1}_{\delta \cdot T_i^{-T}(k+[0,1]^d)}(x) \cdot |(M_{-b_i} f_i)(x) - (M_{-b_i} f_i)(\delta \cdot T_i^{-T} k)|.
\end{aligned}$$

Now, note that $\mathbf{1}_{\delta \cdot T_i^{-T}(k+[0,1]^d)}(x) \neq 0$ implies $\delta \cdot T_i^{-T} k \in x - \delta T_i^{-T}[0,1]^d \subset x + \delta T_i^{-T}[-1,1]^d$. Since we trivially have $x \in x + \delta T_i^{-T}[-1,1]^d$, we obtain

$$|(M_{-b_i} f_i)(x) - (M_{-b_i} f_i)(\delta \cdot T_i^{-T} k)| \leq \left(\text{osc}_{\delta T_i^{-T}[-1,1]^d} [M_{-b_i} f_i] \right)(x).$$

Using again that $\mathbb{R}^d = \biguplus_{k \in \mathbb{Z}^d} \delta T_i^{-T}(k+[0,1]^d)$, we conclude

$$\left| f_i(x) - \left[(\text{Synth}_{\delta,i} \circ \text{Samp}_{\delta,i})(f_j)_{j \in I} \right](x) \right| \leq \left(\text{osc}_{\delta T_i^{-T}[-1,1]^d} [M_{-b_i} f_i] \right)(x) \quad \forall i \in I \quad \forall x \in \mathbb{R}^d \quad \forall (f_j)_{j \in I} \in \Upsilon.$$

Consequently, using the solidity of V_i , we get for

$$\text{Samp}_\delta := \prod_{i \in I} \text{Samp}_{\delta,i} : \Upsilon \rightarrow \left(\mathbb{C}^{\mathbb{Z}^d} \right)^I, (f_j)_{j \in I} \mapsto \left(\text{Samp}_{\delta,i}(f_j)_{j \in I} \right)_{i \in I},$$

$$\text{Synth}_\delta := \bigotimes_{i \in I} \text{Synth}_{\delta,i} : (\mathbb{C}^{\mathbb{Z}^d})^I \rightarrow \left\{ (f_i)_{i \in I} \mid f_i : \mathbb{R}^d \rightarrow \mathbb{C} \text{ measurable } \forall i \in I \right\}, (c_k^{(i)})_{k \in \mathbb{Z}^d, i \in I} \mapsto \left(\text{Synth}_{\delta,i}(c_k^{(i)})_{k \in \mathbb{Z}^d} \right)_{i \in I}$$

that

$$\begin{aligned}
\| (f_i)_{i \in I} - (\text{Synth}_\delta \circ \text{Samp}_\delta)(f_i)_{i \in I} \|_{\ell_w^q([V_i]_{i \in I})} &= \left\| \left(\left\| f_i - \text{Synth}_{\delta,i} \circ \text{Samp}_{\delta,i}(f_j)_{j \in I} \right\|_{V_i} \right)_{i \in I} \right\|_{\ell_w^q} \\
&\leq \left\| \left(\left\| \text{osc}_{\delta T_i^{-T}[-1,1]^d} [M_{-b_i} f_i] \right\|_{V_i} \right)_{i \in I} \right\|_{\ell_w^q} \\
&\stackrel{(\text{eq. (4.14)})}{\leq} \|F_0\| \cdot \delta \cdot \| (f_i)_{i \in I} \|_{\ell_w^q([V_i]_{i \in I})} \quad \forall (f_i)_{i \in I} \in \Upsilon \quad \forall \delta \in (0,1]. \quad (4.16)
\end{aligned}$$

Using the (quasi)-triangle inequality for $\ell_w^q([V_i]_{i \in I})$ (where $2^{\frac{1}{p} + \frac{1}{q}}$ is a valid triangle constant, thanks to [41, Exercise 1.1.5(c)] and to (the proof of) Lemma 4.5), we conclude that

$$T_0^{(\delta)} := \text{Synth}_\delta \circ \text{Samp}_\delta : \Upsilon \rightarrow \ell_w^q([V_i]_{i \in I})$$

is well-defined and bounded, with $\|T_0^{(\delta)}\| \leq 2^{\frac{1}{p} + \frac{1}{q}} \cdot (1 + \|F_0\| \delta) \leq 2^{\frac{1}{p} + \frac{1}{q}} (1 + \|F_0\|)$ for all $\delta \in (0,1]$.

Boundedness of $T_0^{(\delta)}$ —together with estimate (4.16)—is almost sufficient for our purposes, but not quite: In general, it need not be the case that $T_0^{(\delta)}$ maps Υ into Υ . But since Lemma 4.6 shows $F \circ F = F$, it is easy to see $F : \ell_w^q([V_i]_{i \in I}) \rightarrow \Upsilon$, so that $T^{(\delta)} := F \circ T_0^{(\delta)} : \Upsilon \rightarrow \Upsilon$. Furthermore, since $F|_\Upsilon = \text{id}_\Upsilon$, we get

$$\begin{aligned}
\| (f_i)_{i \in I} - T^{(\delta)}(f_i)_{i \in I} \|_{\ell_w^q([V_i]_{i \in I})} &= \| F(f_i)_{i \in I} - F T_0^{(\delta)}(f_i)_{i \in I} \|_{\ell_w^q([V_i]_{i \in I})} \\
&\leq \|F\| \cdot \| (f_i)_{i \in I} - T_0^{(\delta)}(f_i)_{i \in I} \|_{\ell_w^q([V_i]_{i \in I})} \\
&\stackrel{(\text{eq. (4.16)})}{\leq} \|F_0\|^2 \cdot \delta \cdot \| (f_i)_{i \in I} \|_{\ell_w^q([V_i]_{i \in I})} \quad \forall (f_i)_{i \in I} \in \Upsilon \quad \forall \delta \in (0,1]. \quad (4.17)
\end{aligned}$$

But for $0 < \delta \leq \delta_0 = \frac{1}{1+2\|F_0\|^2}$, we have $\|F_0\|^2 \cdot \delta \leq \frac{1}{2}$ and hence $\|\text{id}_\Upsilon - T^{(\delta)}\| \leq \frac{1}{2}$. Using a Neumann-series argument (which is also valid for Quasi-Banach spaces, cf. e.g. [76, Lemma 2.4.11] and thus for the closed subspace Υ of the Quasi-Banach space $\ell_w^q([V_i]_{i \in I})$ thanks to Lemmas 4.5 and 4.6), we conclude that $T^{(\delta)} : \Upsilon \rightarrow \Upsilon$ is boundedly invertible, as long as $0 < \delta \leq \delta_0$.

Now, for arbitrary $f \in \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$, Lemma 4.6 shows that $(\text{Synth}_\mathcal{D} \circ m_\theta \circ \iota \circ \text{Ana}_{\text{osc}})f = f$. The same lemma also shows $F[(\gamma^{(j)} * f)_{j \in I}] = (\gamma^{(j)} * f)_{j \in I}$, i.e., $(\iota \circ \text{Ana}_{\text{osc}})f = (\gamma^{(j)} * f)_{j \in I} \in \Upsilon$. Hence,

$$\begin{aligned} f &= (\text{Synth}_\mathcal{D} \circ m_\theta \circ \iota \circ \text{Ana}_{\text{osc}})f \\ &= \left[(\text{Synth}_\mathcal{D} \circ m_\theta) \circ (T^{(\delta)})^{-1} \circ T^{(\delta)} \circ \iota \circ \text{Ana}_{\text{osc}} \right] f \\ (\text{def. of } T^{(\delta)}) &= \left[\left([\text{Synth}_\mathcal{D} \circ m_\theta] \circ (T^{(\delta)})^{-1} \circ F \circ \text{Synth}_\delta \right) \circ \text{Samp}_\delta \circ \iota \circ \text{Ana}_{\text{osc}} \right] f. \end{aligned}$$

Now, note

$$[(\text{Samp}_\delta \circ \iota \circ \text{Ana}_{\text{osc}})f]_{i,k} = \left(\left[\text{Samp}_\delta (\gamma^{(j)} * f)_{j \in I} \right]_{i,k} \right) = (\gamma^{(i)} * f)(\delta \cdot T_i^{-T}k)$$

and hence $\text{Samp}_\delta \circ \iota \circ \text{Ana}_{\text{osc}} = A_\delta^{(0)}$ on $\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$. Thus, if we define $R_\delta^{(0)} := [\text{Synth}_\mathcal{D} \circ m_\theta] \circ (T^{(\delta)})^{-1} \circ F \circ \text{Synth}_\delta$, we have shown $\text{id}_{\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)} = R_\delta^{(0)} \circ A_\delta^{(0)}$, as claimed. All that remains to show is that $R_\delta^{(0)}, A_\delta^{(0)}$ are indeed well-defined and bounded with domains and codomains as stated at the beginning of the proof.

To this end, note that Lemma 4.4 easily implies that $\text{Synth}_\delta : \ell_{(|\det T_i|^{-1/p}, w_i)}^q([C_i^{(\delta)}]_{i \in I}) \rightarrow \ell_w^q([V_i]_{i \in I})$ is well-defined and bounded. In fact, the lemma even shows that

$$(c_k^{(i)})_{k \in \mathbb{Z}^d, i \in I} \in \ell_{(|\det T_i|^{-1/p}, w_i)}^q([C_i^{(\delta)}]_{i \in I}) \iff \text{Synth}_\delta(c_k^{(i)})_{k \in \mathbb{Z}^d, i \in I} \in \ell_w^q([V_i]_{i \in I})$$

and

$$\left\| \text{Synth}_\delta(c_k^{(i)})_{k \in \mathbb{Z}^d, i \in I} \right\|_{\ell_w^q([V_i]_{i \in I})} \asymp \left\| (c_k^{(i)})_{k \in \mathbb{Z}^d, i \in I} \right\|_{\ell_{(|\det T_i|^{-1/p}, w_i)}^q([C_i^{(\delta)}]_{i \in I})},$$

where the implied constant may depend on δ . Consequently, $R_\delta^{(0)} : \ell_{(|\det T_i|^{-1/p}, w_i)}^q([C_i^{(\delta)}]_{i \in I}) \rightarrow \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$ is indeed well-defined and bounded for $0 < \delta \leq \delta_0$. Furthermore, we see (now for arbitrary $\delta \in (0, 1]$) that

$$\begin{aligned} \|A_\delta^{(0)}f\|_{\ell_{(|\det T_i|^{-1/p}, w_i)}^q([C_i^{(\delta)}]_{i \in I})} &= \|(\text{Samp}_\delta \circ \iota \circ \text{Ana}_{\text{osc}})f\|_{\ell_{(|\det T_i|^{-1/p}, w_i)}^q([C_i^{(\delta)}]_{i \in I})} \\ &\asymp_\delta \|(\text{Synth}_\delta \circ \text{Samp}_\delta \circ \iota \circ \text{Ana}_{\text{osc}})f\|_{\ell_w^q([V_i]_{i \in I})} \\ &= \|(T_0^{(\delta)} \circ \iota \circ \text{Ana}_{\text{osc}})f\|_{\ell_w^q([V_i]_{i \in I})} \\ (\iota \circ \text{Ana}_{\text{osc}} : \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q) \rightarrow \Upsilon, \text{ as seen above}) &\lesssim \|T_0^{(\delta)}\|_{\Upsilon \rightarrow \ell_w^q([V_i]_{i \in I})} \cdot \|\iota \circ \text{Ana}_{\text{osc}}\|_{\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q) \rightarrow \ell_w^q([V_i]_{i \in I})} \cdot \|f\|_{\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)} < \infty \end{aligned}$$

for all $f \in \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$. This finally shows that $A_\delta^{(0)} : \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q) \rightarrow \ell_{(|\det T_i|^{-1/p}, w_i)}^q([C_i^{(\delta)}]_{i \in I})$ is well-defined and bounded for each $\delta \in (0, 1]$ and thus completes the proof of the Banach frame property.

It remains to verify the consistency property stated above. To this end, first define

$$V_j^{(i)} := \begin{cases} L_{v^{(i)}}^{p_i}(\mathbb{R}^d), & \text{if } p_i \in [1, \infty], \\ W_{T_j^{-T}[-1,1]^d}(L_{v^{(i)}}^{p_i}), & \text{if } p_i \in (0, 1), \end{cases}$$

as well as $C_j^{(i, \delta)} = \ell_{(v^{(i)})^{(j, \delta)}}^{p_i}(\mathbb{Z}^d)$ with $(v^{(i)})_k^{(j, \delta)} = v^{(i)}(\delta \cdot T_j^{-T}k)$ for $k \in \mathbb{Z}^d$, $i \in \{1, 2\}$ and $j \in I$. Next, we observe that the domain and codomain of the reconstruction/analysis operators

$$R_\delta^{(0, i)} : \ell_{(|\det T_j|^{-1/p_i}, w_j^{(i)})}^{q_i}([C_j^{(i, \delta)}]_{j \in I}) \rightarrow \mathcal{D}(\mathcal{Q}, L_{v^{(i)}}^{p_i}, \ell_{w^{(i)}}^{q_i})$$

and

$$A_\delta^{(0, i)} : \mathcal{D}(\mathcal{Q}, L_{v^{(i)}}^{p_i}, \ell_{w^{(i)}}^{q_i}) \rightarrow \ell_{(|\det T_j|^{-1/p_i}, w_j^{(i)})}^{q_i}([C_j^{(i, \delta)}]_{j \in I})$$

do depend on $i \in \{1, 2\}$, but the actual *action* of these mappings do not: We always have

$$A_\delta^{(0,i)} f = \left[\left(\gamma^{(j)} * f \right) (\delta \cdot T_j^{-T} k) \right]_{k \in \mathbb{Z}^d, j \in I} \stackrel{\text{eq. (4.8)}}{=} \left[\sum_{\ell \in I} \mathcal{F}^{-1} \left(\widehat{\gamma^{(j)}} \cdot \varphi_\ell \cdot \widehat{f} \right) (\delta \cdot T_j^{-T} k) \right]_{k \in \mathbb{Z}^d, j \in I}$$

and

$$\begin{aligned} R_\delta^{(0,i)} \left(c_k^{(j)} \right)_{k \in \mathbb{Z}^d, j \in I} &= \left([\text{Synth}_{\mathcal{D}} \circ m_\theta] \circ \left(T^{(\delta)} \right)^{-1} \circ F \circ \text{Synth}_\delta \right) \left(c_k^{(j)} \right)_{k \in \mathbb{Z}^d, j \in I} \\ &= \left([\text{Synth}_{\mathcal{D}} \circ m_\theta] \circ \left(T^{(\delta)} \right)^{-1} \circ \iota \circ \text{Ana}_{\text{osc}} \circ \text{Synth}_{\mathcal{D}} \circ m_\theta \circ \text{Synth}_\delta \right) \left(c_k^{(j)} \right)_{k \in \mathbb{Z}^d, j \in I}, \end{aligned}$$

where

$$\begin{aligned} \text{Synth}_{\mathcal{D}} (f_j)_{j \in I} &= \sum_{j \in I} \left[\mathcal{F}^{-1} \left(\varphi_j \cdot \widehat{f_j} \right) \right] \quad \text{with unconditional convergence in } Z'(\mathcal{O}), \\ m_\theta (f_j)_{j \in I} &= \left[(\mathcal{F}^{-1} \theta_j) * f_j \right]_{j \in I}, \\ \iota (f_j)_{j \in I} &= (f_j)_{j \in I}, \\ \text{Ana}_{\text{osc}} f &= \left(\gamma^{(j)} * f \right)_{j \in I} \quad \text{with } \gamma^{(j)} * f \text{ as in equation (4.8),} \\ \text{Synth}_\delta (c_k^{(j)})_{k \in \mathbb{Z}^d, j \in I} &= \left(M_{b_j} \left[\sum_{k \in \mathbb{Z}^d} c_k^{(j)} \cdot e^{-2\pi i \langle b_j, \delta \cdot T_j^{-T} k \rangle} \mathbb{1}_{\delta \cdot T_j^{-T} (k + [0,1]^d)} \right] \right)_{j \in I} \end{aligned}$$

for all $(f_j)_{j \in I} \in \ell_{w^{(i)}}^{q_i}([V_j^{(i)}]_{j \in I})$, all $(c_k^{(j)})_{k \in \mathbb{Z}^d, j \in I} \in \ell_{(|\det T_j|^{-1/p_i} \cdot w_j^{(i)})_{j \in I}}^{q_i}([C_j^{(i,\delta)}]_{j \in I})$, and all $f \in \mathcal{D}(\mathcal{Q}, L_{v^{(i)}}^{p_i}, \ell_{w^{(i)}}^{q_i})$.

Finally, we also have (since $(T^{(\delta)})^{-1}$ can be computed by a Neumann series, as shown above)

$$\left(T^{(\delta)} \right)^{-1} (f_j)_{j \in I} = \left(\text{id} - [\text{id} - T^{(\delta)}] \right)^{-1} (f_j)_{j \in I} = \sum_{n=0}^{\infty} \left(\text{id} - T^{(\delta)} \right)^n (f_j)_{j \in I},$$

where

$$T^{(\delta)} (f_j)_{j \in I} = \left(F \circ T_0^{(\delta)} \right) (f_j)_{j \in I} = (\iota \circ \text{Ana}_{\text{osc}} \circ \text{Synth}_{\mathcal{D}} \circ m_\theta) \circ (\text{Synth}_\delta \circ \text{Samp}_\delta) (f_j)_{j \in I}$$

for

$$(f_j)_{j \in I} \in \Upsilon_i := \left\{ (g_j)_{j \in I} \in \ell_{w^{(i)}}^{q_i}([V_j^{(i)}]_{j \in I}) \mid F(g_j)_{j \in I} = (g_j)_{j \in I} \right\}.$$

In summary, we have shown $R_\delta^{(0,1)} (c_k^{(j)})_{k \in \mathbb{Z}^d, j \in I} = R_\delta^{(0,2)} (c_k^{(j)})_{k \in \mathbb{Z}^d, j \in I}$ and $A_\delta^{(0,1)} f = A_\delta^{(0,2)} f$, as long as both sides of the respective equations are defined. Now, let $f \in \mathcal{D}(\mathcal{Q}, L_{v^{(2)}}^{p_2}, \ell_{w^{(2)}}^{q_2})$ be arbitrary. The implication “ \Rightarrow ” of the consistency statement follows immediately from the main statement of the theorem, so that we only need to show “ \Leftarrow ”. Hence, assume

$$c := A_\delta^{(0,2)} f = \left[\left(\gamma^{(j)} * f \right) (\delta \cdot T_j^{-T} k) \right]_{k \in \mathbb{Z}^d, j \in I} \in \ell_{(|\det T_j|^{-1/p_1} \cdot w_j^{(1)})_{j \in I}}^{q_1}([C_j^{(1,\delta)}]_{j \in I}).$$

We know from above that $f = R_\delta^{(0,2)} A_\delta^{(0,2)} f = R_\delta^{(0,2)} c$. But we have $c \in \ell_{(|\det T_j|^{-1/p_i} \cdot w_j^{(i)})_{j \in I}}^{q_i}([C_j^{(i,\delta)}]_{j \in I})$ for both

$i = 1$ and $i = 2$, so that we get $f = R_\delta^{(0,2)} c = R_\delta^{(0,1)} c$. Since

$$R_\delta^{(0,1)} : \ell_{(|\det T_j|^{-1/p_1} \cdot w_j^{(1)})_{j \in I}}^{q_1}([C_j^{(1,\delta)}]_{j \in I}) \rightarrow \mathcal{D}(\mathcal{Q}, L_{v^{(1)}}^{p_1}, \ell_{w^{(1)}}^{q_1})$$

is well-defined and bounded, we get $f \in \mathcal{D}(\mathcal{Q}, L_{v^{(1)}}^{p_1}, \ell_{w^{(1)}}^{q_1})$, as claimed. \square

The main limitation of Theorem 4.7 is its somewhat opaque set of assumptions regarding $\Gamma = (\gamma_i)_{i \in I}$. In Section 6 (see in particular Corollary 6.6), we will derive more transparent criteria which ensure that Theorem 4.7 is applicable.

But before that, we first consider the “dual” problem to the Banach frame property, i.e., we show that the family $\left(L_{\delta \cdot T_i^{-T} k} \gamma^{[i]} \right)_{k \in \mathbb{Z}^d, i \in I}$ forms an *atomic decomposition* for the decomposition space $\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$, under suitable assumptions on $\Gamma = (\gamma_i)_{i \in I}$. Proving this is the main goal of the next section.

5. ATOMIC DECOMPOSITIONS

In this section, we show the dual statement to the preceding section, i.e., we show that the (discretely translated) $\gamma^{[j]}$ not only form a Banach frame for $\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$, but also an atomic decomposition. For this, we introduce still another set of assumptions:

Assumption 5.1. We assume that for each $i \in I$, we are given functions $\gamma_i, \gamma_{i,1}, \gamma_{i,2}$ with the following properties:

- (1) We have $\gamma_{i,1}, \gamma_{i,2} \in L_{(1+|\bullet|)^\kappa}^1(\mathbb{R}^d) \hookrightarrow L_{v_0}^1(\mathbb{R}^d) \hookrightarrow L^1(\mathbb{R}^d)$ for all $i \in I$.
- (2) We have $\gamma_i = \gamma_{i,1} * \gamma_{i,2}$ for all $i \in I$.
- (3) We have $\widehat{\gamma_{i,1}}, \widehat{\gamma_{i,2}} \in C^\infty(\mathbb{R}^d)$ for all $i \in I$ and all partial derivatives of $\widehat{\gamma_{i,1}}, \widehat{\gamma_{i,2}}$ have at most polynomial growth.
- (4) We have $\gamma_{i,2} \in C^1(\mathbb{R}^d)$ with $\nabla \gamma_{i,2} \in L_{v_0}^1(\mathbb{R}^d)$ for all $i \in I$.
- (5) The constant

$$\Omega_4^{(p,K)} := \sup_{i \in I} \|\gamma_{i,2}\|_{K_0} + \sup_{i \in I} \|\nabla \gamma_{i,2}\|_{K_0} \quad (5.1)$$

is finite. Here, $K_0 := K + \frac{d}{\min\{1,p\}} + 1$ and

$$\|f\|_{K_0} := \sup_{x \in \mathbb{R}^d} (1 + |x|)^{K_0} |f(x)| \in [0, \infty].$$

- (6) We have $\|\gamma_i\|_{K_0} < \infty$ for all $i \in I$.
- (7) For $\ell \in \{1, 2\}$ and $i \in I$, define

$$\gamma_\ell^{(i)} := \mathcal{F}^{-1}(\widehat{\gamma_{i,\ell}} \circ S_i^{-1}) = |\det T_i| \cdot M_{b_i}[\gamma_{i,\ell} \circ T_i^T], \quad (5.2)$$

so that $\gamma_\ell^{(i)}$ is to $\gamma_{i,\ell}$ as $\gamma^{(i)}$ is to γ_i .

- (8) For $i, j \in I$ set

$$C_{i,j} := \begin{cases} \left\| \mathcal{F}^{-1} \left(\varphi_i \cdot \widehat{\gamma_1^{(j)}} \right) \right\|_{L_{v_0}^1}, & \text{if } p \in [1, \infty], \\ (1 + \|T_j^{-1} T_i\|)^{pK+d} \cdot |\det T_j|^{1-p} \cdot \left\| \mathcal{F}^{-1} \left(\varphi_i \cdot \widehat{\gamma_1^{(j)}} \right) \right\|_{L_{v_0}^p}^p, & \text{if } p \in (0, 1). \end{cases} \quad (5.3)$$

- (9) With $r := \max\left\{q, \frac{q}{p}\right\}$, we assume that the operator \vec{C} induced by $(C_{i,j})_{i,j \in I}$, i.e.

$$\vec{C}(c_j)_{j \in I} := \left(\sum_{j \in I} C_{i,j} c_j \right)_{i \in I}$$

defines a well-defined, bounded operator $\vec{C} : \ell_{w^{\min\{1,p\}}}^r(I) \rightarrow \ell_{w^{\min\{1,p\}}}^r(I)$. ◀

Remark. In the following, we will often use Γ_ℓ as a short notation for the family $\Gamma_\ell = (\gamma_{i,\ell})_{i \in I}$ ($\ell \in \{1, 2\}$), similar to the notation $\Gamma = (\gamma_i)_{i \in I}$.

The assumptions above are slightly redundant. In particular, since $v_0(x) \leq \Omega_1 \cdot (1 + |x|)^K$, it is an easy consequence of equation (1.9) that $\|\gamma_{i,2}\|_{K_0} < \infty$ and $\|\nabla \gamma_{i,2}\|_{K_0} < \infty$ already imply $\gamma_{i,2} \in L_{(1+|\bullet|)^\kappa}^1(\mathbb{R}^d) \hookrightarrow L_{v_0}^1(\mathbb{R}^d)$ and $\nabla \gamma_{i,2} \in L_{(1+|\bullet|)^\kappa}^1(\mathbb{R}^d) \hookrightarrow L_{v_0}^1(\mathbb{R}^d)$, respectively.

Exactly as in Remark 3.2, we see that $\gamma_{i,1}, \gamma_{i,2} \in L_{(1+|\bullet|)^\kappa}^1(\mathbb{R}^d)$ entails $\gamma_1^{(j)}, \gamma_2^{(j)} \in L_{(1+|\bullet|)^\kappa}^1(\mathbb{R}^d) \hookrightarrow L_{v_0}^1(\mathbb{R}^d)$ for all $j \in I$. ◆

Part of the definition of an atomic decomposition $(\theta_\ell)_{\ell \in L}$ is that the synthesis map $(c_\ell)_{\ell \in L} \mapsto \sum_{\ell \in L} c_\ell \theta_\ell$ is bounded, when defined on a suitable sequence space. Our next lemma establishes a variant of this property for a certain *continuous* (as opposed to discrete) synthesis operator. This lemma should be compared to Lemma 3.9.

Lemma 5.2. Assume that the family $\Gamma_1 = (\gamma_{i,1})_{i \in I}$ satisfies Assumption 5.1. Then, the operator

$$\text{Synth}_{\Gamma_1} : \ell_w^q \left([V_j]_{j \in I} \right) \rightarrow \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q) \leq Z'(\mathcal{O}), (g_j)_{j \in I} \mapsto \sum_{j \in I} \gamma_1^{(j)} * g_j \stackrel{\text{Lem. 3.3}}{=} \sum_{j \in I} \mathcal{F}^{-1} \left(\widehat{\gamma_1^{(j)}} \cdot \widehat{g_j} \right)$$

is well-defined and bounded with

$$\|\text{Synth}_{\Gamma_1}\| \leq C \cdot \|\vec{C}\|^{\max\{1, \frac{1}{p}\}} \text{ with } C = \begin{cases} 1, & \text{if } p \geq 1 \\ \frac{(2^6/\sqrt{d})^{\frac{d}{p}}}{2^{21} \cdot d^7} \cdot \left(2^{21} \cdot d^5 \cdot \left\lceil K + \frac{d+1}{p} \right\rceil\right)^{\left\lceil K + \frac{d+1}{p} \right\rceil + 1} \cdot (1+R_Q)^{\frac{d}{p}} \cdot \Omega_0^{5K} \Omega_1^5, & \text{if } p < 1. \end{cases}$$

Here, $\text{Synth}_{\Gamma_1}(g_j)_{j \in I}$ is the linear functional

$$Z(\mathcal{O}) \rightarrow \mathbb{C}, f \mapsto \sum_{j \in I} \left\langle \widehat{\gamma_1^{(j)}} \cdot \widehat{g_j}, \mathcal{F}^{-1} f \right\rangle_{\mathcal{S}', \mathcal{S}} = \sum_{j \in I} \left\langle \mathcal{F}^{-1} \left(\widehat{\gamma_1^{(j)}} \cdot \widehat{g_j} \right), f \right\rangle_{\mathcal{S}', \mathcal{S}} \stackrel{\text{Lem. 3.3}}{=} \sum_{j \in I} \left\langle \gamma_1^{(j)} * g_j, f \right\rangle_{\mathcal{S}', \mathcal{S}}, \quad (5.4)$$

where each of the series converges absolutely for each $f \in Z(\mathcal{O})$. \blacktriangleleft

Proof. First of all, recall from Lemma 3.3 that $V_j \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$ for all $j \in I$. Thus, for $(g_j)_{j \in I} \in \ell_w^q([V_j]_{j \in I})$, we see that $\widehat{g_j} \in \mathcal{S}'(\mathbb{R}^d)$ is a well-defined tempered distribution for all $j \in I$. In view of the inclusion $Z(\mathcal{O}) \hookrightarrow \mathcal{S}(\mathbb{R}^d)$, we thus see that every *individual* term of each of the series in equation (5.4) is well-defined. Here, we use that $\widehat{\gamma_{j,1}} \in C^\infty(\mathbb{R}^d)$ with all derivatives of at most polynomial growth, so that the same holds for $\widehat{\gamma_1^{(j)}} = \widehat{\gamma_{j,1}} \circ S_j^{-1}$. We still have to show, however, that (each of) the series in equation (5.4) converges (absolutely) for every $f \in Z(\mathcal{O})$ and defines a continuous linear functional.

Since $Z(\mathcal{O}) = \mathcal{F}(C_c^\infty(\mathcal{O}))$, this is equivalent to showing for arbitrary $(g_j)_{j \in I} \in \ell_w^q([V_j]_{j \in I})$ that the series defining the functional

$$\phi : C_c^\infty(\mathcal{O}) \rightarrow \mathbb{C}, f \mapsto \sum_{j \in I} \left\langle \widehat{\gamma_1^{(j)}} \cdot \widehat{g_j}, f \right\rangle_{\mathcal{S}', \mathcal{S}}$$

converges absolutely for each $f \in C_c^\infty(\mathcal{O})$ and that $\phi \in \mathcal{D}'(\mathcal{O})$. With the same reasoning as above, we see at least that each term in the series is well-defined. In the following, we will show that $\phi \in \mathcal{D}'(\mathcal{O})$ is indeed well-defined, with absolute convergence of the series.

But first, let us *assume* that this is the case. Then we have, for fixed $i \in I$ (by the usual formula for the (inverse) Fourier transform of a compactly supported distribution, see e.g. [70, Theorem 7.23])

$$\begin{aligned} |[\mathcal{F}^{-1}(\varphi_i \phi)](x)| &= \left| \left\langle \phi, \varphi_i \cdot e^{2\pi i \langle x, \cdot \rangle} \right\rangle_{\mathcal{D}'(\mathcal{O}), C_c^\infty(\mathcal{O})} \right| \\ &= \left| \sum_{j \in I} \left\langle \widehat{\gamma_1^{(j)}} \cdot \widehat{g_j}, \varphi_i \cdot e^{2\pi i \langle x, \cdot \rangle} \right\rangle_{\mathcal{S}', \mathcal{S}} \right| \\ &\leq \sum_{j \in I} \left| \left\langle \widehat{\gamma_1^{(j)}} \cdot \widehat{g_j}, \varphi_i \cdot e^{2\pi i \langle x, \cdot \rangle} \right\rangle_{\mathcal{S}', \mathcal{S}} \right| \\ &= \sum_{j \in I} \left| \left[\mathcal{F}^{-1} \left(\varphi_i \cdot \widehat{\gamma_1^{(j)}} \cdot \widehat{g_j} \right) \right](x) \right| \\ &\stackrel{(\text{Lemma 3.3})}{=} \sum_{j \in I} \left| \left[\mathcal{F}^{-1} \left(\varphi_i \cdot \widehat{\gamma_1^{(j)}} \right) * g_j \right](x) \right| \quad \forall x \in \mathbb{R}^d, \end{aligned} \quad (5.5)$$

where all but the first three terms always make sense (as elements of $[0, \infty]$), even without assuming that ϕ is a well-defined distribution.

Now, we invoke Theorem 2.17 to obtain for all $f \in \mathcal{S}'(\mathbb{R}^d)$ with $\text{supp } \widehat{f} \subset \overline{Q_i} \subset T_i[-R_Q, R_Q]^d + b_i$ that $\|f\|_{W_{T_i^{-T}[-1,1]^d}(L_v^p)} \lesssim \|f\|_{L_v^p}$, where the implied constant depends on p, d and on K, R_Q , which are fixed throughout. In combination with the embedding $W_{T_i^{-T}[-1,1]^d}(L_v^p) \hookrightarrow L_v^\infty(\mathbb{R}^d) \hookrightarrow L_{(1+|\bullet|)^{-K}}^\infty(\mathbb{R}^d)$ from equation (2.13) (where now the norm of the embedding depends on i (and on p, d, K, v)), we thus get for every $i \in I$ some constant $C^{(i)} = C^{(i)}(p, d, K, v, R_Q) > 0$ such that $\|f\|_* \leq C^{(i)} \cdot \|f\|_{L_v^p}$ for all $f \in \mathcal{S}'(\mathbb{R}^d)$ satisfying $\text{supp } \widehat{f} \subset \overline{Q_i}$, with $\|f\|_* := \sup_{x \in \mathbb{R}^d} (1 + |x|)^{-K} |f(x)|$. Since $\text{supp } \mathcal{F} \left[\mathcal{F}^{-1} \left(\varphi_i \cdot \widehat{\gamma_1^{(j)}} \right) * g_j \right] = \text{supp} \left(\varphi_i \cdot \widehat{\gamma_1^{(j)}} \cdot \widehat{g_j} \right) \subset \overline{Q_i}$, this yields

$$\left\| \mathcal{F}^{-1} \left(\varphi_i \cdot \widehat{\gamma_1^{(j)}} \right) * g_j \right\|_* \leq C^{(i)} \cdot \left\| \mathcal{F}^{-1} \left(\varphi_i \cdot \widehat{\gamma_1^{(j)}} \right) * g_j \right\|_{L_v^p} \quad \forall j \in I.$$

Now, we distinguish two cases:

Case 1: We have $p \in [1, \infty]$. In this case, we can simply use the weighted Young inequality (equation (1.12)) to derive

$$\left\| \mathcal{F}^{-1} \left(\varphi_i \cdot \widehat{\gamma_1^{(j)}} \right) * g_j \right\|_{L_v^p} \leq \left\| \mathcal{F}^{-1} \left(\varphi_i \cdot \widehat{\gamma_1^{(j)}} \right) \right\|_{L_{v_0}^1} \cdot \|g_j\|_{L_v^p} = C_{i,j} \cdot \|g_j\|_{V_j}.$$

But since we have $c = (c_j)_{j \in I} \in \ell_w^q(I) = \ell_{w^{\min\{1,p\}}}^r(I)$ for $c_j := \|g_j\|_{V_j}$, we get by boundedness of \vec{C} that $\vec{C}c \in \ell_w^q(I)$. In particular,

$$\frac{1}{C^{(i)}} \cdot \sum_{j \in I} \left\| \mathcal{F}^{-1} \left(\varphi_i \cdot \widehat{\gamma_1^{(j)}} \right) * g_j \right\|_* \leq \sum_{j \in I} \left\| \mathcal{F}^{-1} \left(\varphi_i \cdot \widehat{\gamma_1^{(j)}} \right) * g_j \right\|_{L_v^p} \leq \sum_{j \in I} C_{i,j} \|g_j\|_{V_j} = (\vec{C}c)_i < \infty, \quad (5.6)$$

from which it follows that the series $\sum_{j \in I} \left[\mathcal{F}^{-1} \left(\varphi_i \cdot \widehat{\gamma_1^{(j)}} \right) * g_j \right] (x)$ converges absolutely for all $x \in \mathbb{R}^d$ (even locally uniformly in x).

Furthermore, for arbitrary $\theta \in C_c^\infty(\mathcal{O})$, we have

$$\begin{aligned} \sum_{j \in I} \left| \left\langle \widehat{\gamma_1^{(j)}} \cdot \widehat{g_j}, \varphi_i \theta \right\rangle_{S', S} \right| &= \sum_{j \in I} \left| \left\langle \mathcal{F}^{-1} \left(\varphi_i \cdot \widehat{\gamma_1^{(j)}} \right) \cdot \widehat{g_j}, \widehat{\theta} \right\rangle_{S', S} \right| \\ &\leq \sum_{j \in I} \left\| \mathcal{F}^{-1} \left(\varphi_i \cdot \widehat{\gamma_1^{(j)}} \right) * g_j \right\|_* \|\widehat{\theta}\|_{L^1_{(1+|\bullet|)^K}} \leq C^{(i)} \cdot (\vec{C}c)_i \cdot \|\widehat{\theta}\|_{L^1_{(1+|\bullet|)^K}} < \infty, \end{aligned}$$

so that the series $\sum_{j \in I} \left\langle \widehat{\gamma_1^{(j)}} \cdot \widehat{g_j}, \varphi_i \theta \right\rangle$ defining $\phi(\varphi_i \theta)$ converges absolutely. The same estimate also shows that $\theta \mapsto \phi(\varphi_i \theta)$ is a distribution on \mathcal{O} , since $\theta \mapsto \|\widehat{\theta}\|_{L^1_{(1+|\bullet|)^K}}$ is a continuous seminorm on $C_c^\infty(\mathcal{O}) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$.

But since $(\varphi_i)_{i \in I}$ is a locally finite partition of unity on \mathcal{O} , we have $\theta = \sum_{i \in I_\Upsilon} \varphi_i \theta$ for every $\theta \in C_c^\infty(\mathcal{O})$ with $\text{supp } \theta \subset \Upsilon$, where $\Upsilon \subset \mathcal{O}$ is an arbitrary compact set and where $I_\Upsilon \subset I$ is finite. Hence, $\theta \mapsto \phi(\theta) = \sum_{i \in I_\Upsilon} \phi(\varphi_i \theta)$ is a continuous linear functional on $\{\theta \in C_c^\infty(\mathcal{O}) \mid \text{supp } \theta \subset \Upsilon\}$ for arbitrary compact $\Upsilon \subset \mathcal{O}$ and the defining series converges absolutely (as a *finite* sum of absolutely convergent series). This shows that $\phi \in \mathcal{D}'(\mathcal{O})$ is well-defined (with absolute convergence of the defining series), so that equation (5.5) is valid.

As a consequence of equations (5.5) and (5.6) and of the triangle inequality for $L_v^p(\mathbb{R}^d)$, we finally get

$$\left\| \mathcal{F}^{-1}(\varphi_i \phi) \right\|_{L_v^p} \leq \sum_{j \in I} \left\| \mathcal{F}^{-1} \left(\varphi_i \cdot \widehat{\gamma_1^{(j)}} \right) * g_j \right\|_{L_v^p} \leq (\vec{C}c)_i \quad \forall i \in I,$$

so that solidity of $\ell_w^q(I)$ yields

$$\|\phi\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L_v^p, \ell_w^q)} = \left\| \left(\left\| \mathcal{F}^{-1}(\varphi_i \phi) \right\|_{L_v^p} \right)_{i \in I} \right\|_{\ell_w^q} \leq \|\vec{C}c\|_{\ell_w^q} \leq \|\vec{C}\| \cdot \|c\|_{\ell_w^q} = \|\vec{C}\| \cdot \left\| (g_j)_{j \in I} \right\|_{\ell_w^q([V_i]_{i \in I})} < \infty.$$

All in all, we see that $\phi \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L_v^p, \ell_w^q) \leq \mathcal{D}'(\mathcal{O})$ is well-defined. But by definition of Synth_{Γ_1} , we have $\text{Synth}_{\Gamma_1}(g_j)_{j \in I} = \mathcal{F}^{-1}\phi$ for the isometric isomorphism $\mathcal{F}^{-1} : \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L_v^p, \ell_w^q) \rightarrow \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q), \psi \mapsto \psi \circ \mathcal{F}^{-1}$. As a consequence, $\text{Synth}_{\Gamma_1}(g_j)_{j \in I} \in \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$ is well-defined and

$$\left\| \text{Synth}_{\Gamma_1}(g_j)_{j \in I} \right\|_{\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)} = \|\phi\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L_v^p, \ell_w^q)} \leq \|\vec{C}\| \cdot \left\| (g_j)_{j \in I} \right\|_{\ell_w^q([V_i]_{i \in I})},$$

as desired.

Case 2: We have $p \in (0, 1)$. In this case, we replace the application of the weighted Young inequality (equation (1.12)) by an application of Corollary 2.22 to get for $C_1 := d^{-\frac{d}{2p}} \cdot (972 \cdot d^{5/2})^{K+\frac{d}{p}} \cdot \Omega_0^{3K} \Omega_1^3$ that

$$\begin{aligned}
& \left\| \mathcal{F}^{-1} \left(\varphi_i \cdot \widehat{\gamma_1^{(j)}} \right) * g_j \right\|_{L_v^p} \\
& \stackrel{(\text{Lemma 2.2})}{\leq} \left\| \mathcal{F}^{-1} \left(\varphi_i \cdot \widehat{\gamma_1^{(j)}} \right) * g_j \right\|_{W_{T_j^{-T}[-1,1]^d}(L_v^p)} \\
& \stackrel{(\text{Cor. 2.22})}{\leq} C_1 \cdot |\det T_j|^{\frac{1}{p}-1} \cdot \left\| \mathcal{F}^{-1} \left(\varphi_i \cdot \widehat{\gamma_1^{(j)}} \right) \right\|_{W_{T_j^{-T}[-1,1]^d}(L_{v_0}^p)} \cdot \|g_j\|_{W_{T_j^{-T}[-1,1]^d}(L_{v_0}^p)} \\
& \stackrel{(\text{eq. (2.3)})}{\leq} C_1 (6d)^{K+\frac{d}{p}} \cdot \Omega_0^K \Omega_1 \cdot (1 + \|T_j^{-1} T_i\|)^{K+\frac{d}{p}} \cdot |\det T_j|^{\frac{1}{p}-1} \cdot \left\| \mathcal{F}^{-1} \left(\varphi_i \cdot \widehat{\gamma_1^{(j)}} \right) \right\|_{W_{T_i^{-T}[-1,1]^d}(L_{v_0}^p)} \cdot \|g_j\|_{V_j} \\
& \stackrel{(\dagger)}{\stackrel{(\text{Thm. 2.17})}{\leq}} C_1 C_2 (6d)^{K+\frac{d}{p}} \cdot \Omega_0^K \Omega_1 \cdot (1 + \|T_j^{-1} T_i\|)^{K+\frac{d}{p}} \cdot |\det T_j|^{\frac{1}{p}-1} \cdot \left\| \mathcal{F}^{-1} \left(\varphi_i \cdot \widehat{\gamma_1^{(j)}} \right) \right\|_{L_{v_0}^p} \cdot \|g_j\|_{V_j} \\
& = C_1 C_2 (6d)^{K+\frac{d}{p}} \cdot \Omega_0^K \Omega_1 \cdot C_{i,j}^{1/p} \cdot \|g_j\|_{V_j} =: C_3 \cdot C_{i,j}^{1/p} \cdot \|g_j\|_{V_j}, \tag{5.7}
\end{aligned}$$

where the step marked with (\dagger) used that

$$\text{supp} \left(\varphi_i \cdot \widehat{\gamma_1^{(j)}} \right) \subset \overline{Q_i} \subset T_i [\overline{B_{R_Q}}(0)] + b_i \subset T_i [-R_Q, R_Q]^d + b_i,$$

so that Theorem 2.17 (with v_0 instead of v) yields $\left\| \mathcal{F}^{-1} \left(\varphi_i \cdot \widehat{\gamma_1^{(j)}} \right) \right\|_{W_{T_i^{-T}[-1,1]^d}(L_{v_0}^p)} \leq C_2 \cdot \left\| \mathcal{F}^{-1} \left(\varphi_i \cdot \widehat{\gamma_1^{(j)}} \right) \right\|_{L_{v_0}^p}$ for

$$C_2 := 2^{4(1+\frac{d}{p})} s_d^{\frac{1}{p}} \left(192 \cdot d^{3/2} \cdot \left\lceil K + \frac{d+1}{p} \right\rceil \right)^{\lceil K+\frac{d+1}{p} \rceil + 1} \cdot \Omega_0^K \Omega_1 \cdot (1 + R_Q)^{\frac{d}{p}}.$$

Next, set $c_j := \|g_j\|_{V_j}^p$ for $j \in I$ and note that $(g_j)_{j \in I} \in \ell_w^q([V_j]_{j \in I})$ yields $c = (c_j)_{j \in I} \in \ell_{w^p}^{q/p}(I) = \ell_{w^{\min\{1,p\}}}^r(I)$. Hence, we get because of $\ell^p \hookrightarrow \ell^1$ that

$$\begin{aligned}
\frac{1}{C^{(i)}} \cdot \sum_{j \in I} \left\| \mathcal{F}^{-1} \left(\varphi_i \cdot \widehat{\gamma_1^{(j)}} \right) * g_j \right\|_* & \leq \left(\sum_{j \in I} \left\| \mathcal{F}^{-1} \left(\varphi_i \cdot \widehat{\gamma_1^{(j)}} \right) * g_j \right\|_{L_v^p}^p \right)^{1/p} \\
& \leq C_3 \cdot \left(\sum_{j \in I} C_{i,j} \cdot c_j \right)^{1/p} \\
& = C_3 \cdot (\vec{C} \cdot c)_i^{1/p} < \infty,
\end{aligned}$$

which is a slight variation of equation (5.6). Now, we see exactly as in case of $p \in [1, \infty]$ that ϕ is a well-defined distribution $\phi \in \mathcal{D}'(\mathcal{O})$, so that equation (5.5) is valid.

Using this equation and the p -triangle inequality for $L_v^p(\mathbb{R}^d)$, we derive

$$\begin{aligned}
\left\| \mathcal{F}^{-1}(\varphi_i \phi) \right\|_{L_v^p} & \leq \left(\sum_{j \in I} \left\| \mathcal{F}^{-1} \left(\varphi_i \cdot \widehat{\gamma_1^{(j)}} \right) * g_j \right\|_{L_v^p}^p \right)^{1/p} \\
& \stackrel{(\text{eq. (5.7)})}{\leq} C_3 \cdot \left(\sum_{j \in I} C_{i,j} \cdot \|g_j\|_{V_j}^p \right)^{1/p} \\
& = C_3 \cdot (\vec{C} \cdot c)_i^{1/p} < \infty
\end{aligned}$$

and hence

$$\begin{aligned}
\|\phi\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L_v^p, \ell_w^q)} &= \left\| \left(\|\mathcal{F}^{-1}(\varphi_i \phi)\|_{L_v^p} \right)_{i \in I} \right\|_{\ell_w^q} \\
&\leq C_3 \cdot \|(\vec{C} \cdot c)^{1/p}\|_{\ell_w^q} \\
&= C_3 \cdot \|w^p \cdot [\vec{C} \cdot c]\|_{\ell^{q/p}}^{1/p} \\
&= C_3 \cdot \|\vec{C} \cdot c\|_{\ell^{r \min\{1, p\}}}^{1/p} \\
&\leq C_3 \cdot \|\vec{C}\|^{1/p} \cdot \|c\|_{\ell^{r \min\{1, p\}}}^{1/p} \\
&= C_3 \cdot \|\vec{C}\|^{1/p} \cdot \left\| \left(w_j \cdot \|g_j\|_{V_j} \right)_{j \in I} \right\|_{\ell^q} \\
&= C_3 \cdot \|\vec{C}\|^{1/p} \cdot \left\| (g_j)_{j \in I} \right\|_{\ell_w^q([V_j]_{j \in I})} < \infty.
\end{aligned}$$

Now, we see as for $p \in [1, \infty]$ that Synth_{Γ_1} is a bounded linear operator with $\|\text{Synth}_{\Gamma_1}\| \leq C_3 \cdot \|\vec{C}\|^{1/p}$.

Finally, we observe, using $s_d \leq 2^{2d}$, that

$$\begin{aligned}
C_3 &= C_1 C_2 (6d)^{K + \frac{d}{p}} \cdot \Omega_0^K \Omega_1 \\
&= d^{-\frac{d}{2p}} \cdot \left(972 \cdot d^{5/2} \right)^{K + \frac{d}{p}} (6d)^{K + \frac{d}{p}} \cdot 2^{4(1 + \frac{d}{p})} s_d^{\frac{1}{p}} \left(192 \cdot d^{3/2} \cdot \left\lceil K + \frac{d+1}{p} \right\rceil \right)^{\lceil K + \frac{d+1}{p} \rceil + 1} \cdot (1 + R_{\mathcal{Q}})^{\frac{d}{p}} \cdot \Omega_0^{5K} \Omega_1^5 \\
&\leq d^{-\frac{d}{2p}} \cdot \left(5832 \cdot d^{7/2} \right)^{K + \frac{d}{p}} \cdot 2^{4+6\frac{d}{p}} \left(192 \cdot d^{3/2} \cdot \left\lceil K + \frac{d+1}{p} \right\rceil \right)^{\lceil K + \frac{d+1}{p} \rceil + 1} \cdot (1 + R_{\mathcal{Q}})^{\frac{d}{p}} \cdot \Omega_0^{5K} \Omega_1^5 \\
&\leq d^{-\frac{d}{2p}} \cdot \left(5832 \cdot d^{7/2} \right)^{-2} \cdot 2^{4+6\frac{d}{p}} \left(2^{21} \cdot d^5 \cdot \left\lceil K + \frac{d+1}{p} \right\rceil \right)^{\lceil K + \frac{d+1}{p} \rceil + 1} \cdot (1 + R_{\mathcal{Q}})^{\frac{d}{p}} \cdot \Omega_0^{5K} \Omega_1^5 \\
&\leq \frac{(2^6/\sqrt{d})^{d/p}}{2^{21} \cdot d^7} \cdot \left(2^{21} \cdot d^5 \cdot \left\lceil K + \frac{d+1}{p} \right\rceil \right)^{\lceil K + \frac{d+1}{p} \rceil + 1} \cdot (1 + R_{\mathcal{Q}})^{\frac{d}{p}} \cdot \Omega_0^{5K} \Omega_1^5,
\end{aligned}$$

which completes the proof. \square

In order to switch from the continuous synthesis operator from the preceding lemma to a discrete one, our next technical result is helpful.

Lemma 5.3. *Let $p \in (0, \infty]$ and assume that $\varrho : \mathbb{R}^d \rightarrow \mathbb{C}$ is measurable and satisfies $\|\varrho\|_{K_0} < \infty$ with K_0 and $\|\bullet\|_{K_0}$ as in Assumption 5.1. Let $i \in I$ and $\delta \in (0, 1]$ and let V_i be defined as in Assumption 3.1. Furthermore, let the **coefficient space** $C_i^{(\delta)}$ be defined as in equation (4.2).*

Then, the maps

$$\Psi_{|\varrho|}^{(i, \delta)} : C_i^{(\delta)} \rightarrow V_i, (c_k)_{k \in \mathbb{Z}^d} \mapsto \left(\sum_{k \in \mathbb{Z}^d} c_k \cdot L_{\delta \cdot k} |\varrho| \right) \circ T_i^T = \sum_{k \in \mathbb{Z}^d} c_k \cdot L_{\delta \cdot T_i^{-T} k} |\varrho \circ T_i^T|$$

and

$$\Psi_{\varrho}^{(i, \delta)} : C_i^{(\delta)} \rightarrow V_i, (c_k)_{k \in \mathbb{Z}^d} \mapsto \left(\sum_{k \in \mathbb{Z}^d} c_k \cdot L_{\delta \cdot k} \varrho \right) \circ T_i^T = \sum_{k \in \mathbb{Z}^d} c_k \cdot L_{\delta \cdot T_i^{-T} k} [\varrho \circ T_i^T]$$

are well-defined and bounded, with pointwise absolute convergence of the series and with

$$\left\| \Psi_{\varrho}^{(i, \delta)} \right\| \leq \left\| \Psi_{|\varrho|}^{(i, \delta)} \right\| \leq \begin{cases} \Omega_0^K \Omega_1 \cdot (1 + 2\sqrt{d})^{K_0} \left(\frac{s_d}{p} \right)^{1/p} \cdot \|\varrho\|_{K_0} \cdot |\det T_i|^{-\frac{1}{p}}, & \text{if } p < 1, \\ \Omega_0^K \Omega_1 \cdot 12^{d+1} \cdot \delta^{-(1 - \frac{1}{p})(d+1)} \cdot \|\varrho\|_{K_0} \cdot |\det T_i|^{-\frac{1}{p}}, & \text{if } p \geq 1. \end{cases}$$

In particular, if $g \in L_{v_0}^1(\mathbb{R}^d)$, then

$$g * \left[\Psi_{\varrho}^{(i, \delta)} (c_k)_{k \in \mathbb{Z}^d} \right] = \sum_{k \in \mathbb{Z}^d} \left(c_k \cdot \left[g * L_{\delta \cdot T_i^{-T} k} [\varrho \circ T_i^T] \right] \right). \quad (5.8)$$

◀

Proof. Clearly, since V_i and $C_i^{(\delta)}$ are solid, boundedness of $\Psi_{|\varrho|}^{(i,\delta)}$ implies that of $\Psi_{\varrho}^{(i,\delta)}$, with $\|\Psi_{\varrho}^{(i,\delta)}\| \leq \|\Psi_{|\varrho|}^{(i,\delta)}\|$. Furthermore, again by solidity and since $|\varrho(x)| \leq \|\varrho\|_{K_0} \cdot (1 + |x|)^{-K_0}$ for all $x \in \mathbb{R}^d$, it suffices to prove the claim (except for equation (5.8)) for the special case $\varrho(x) = (1 + |x|)^{-K_0}$, so that $\|\varrho\|_{K_0} = 1$.

Recall from equation (4.2) that $v_k^{(j,\delta)} = v(\delta \cdot T_j^{-T} k)$. Now, we first observe

$$v(x) = v(\delta \cdot T_i^{-T} k + x - \delta \cdot T_i^{-T} k) \leq v(\delta \cdot T_i^{-T} k) \cdot v_0(x - \delta \cdot T_i^{-T} k) = v_k^{(i,\delta)} \cdot v_0(x - \delta \cdot T_i^{-T} k), \quad (5.9)$$

so that

$$0 \leq \frac{v(x)}{v_k^{(i,\delta)}} \cdot \left(L_{\delta \cdot T_i^{-T} k} [\varrho \circ T_i^T] \right)(x) \leq v_0(x - \delta \cdot T_i^{-T} k) \cdot (\varrho \circ T_i^T)(x - \delta \cdot T_i^{-T} k).$$

By translation invariance of $\|\bullet\|_{L^1}$, this implies for $p \in [1, \infty]$ (which entails $K_0 = K + d + 1$) that

$$\begin{aligned} \left\| \frac{L_{\delta \cdot T_i^{-T} k} [\varrho \circ T_i^T]}{v_k^{(i,\delta)}} \right\|_{L_v^1} &\leq \|x \mapsto v_0(x - \delta \cdot T_i^{-T} k) \cdot (\varrho \circ T_i^T)(x - \delta \cdot T_i^{-T} k)\|_{L^1} \\ &= \|v_0 \cdot (\varrho \circ T_i^T)\|_{L^1} \\ (\text{standard change of variables}) &= |\det T_i^T|^{-1} \cdot \|(v_0 \circ T_i^{-T}) \cdot \varrho\|_{L^1} \\ (\text{assumptions on } v_0) &\leq \Omega_1 \cdot |\det T_i|^{-1} \cdot \|x \mapsto (1 + |T_i^{-T} x|)^K \cdot \varrho(x)\|_{L^1} \\ (\text{eq. (1.11)}) &\leq \Omega_0^K \Omega_1 \cdot |\det T_i|^{-1} \cdot \|x \mapsto (1 + |x|)^K \cdot \varrho(x)\|_{L^1} \\ (K_0 = K + d + 1 \text{ since } p \in [1, \infty]) &= \Omega_0^K \Omega_1 \cdot |\det T_i|^{-1} \cdot \|x \mapsto (1 + |x|)^{-(d+1)}\|_{L^1} \\ (\text{eq. (1.9)}) &\leq \Omega_0^K \Omega_1 s_d \cdot |\det T_i|^{-1}. \end{aligned}$$

Hence, we get in case of $p = 1$ that

$$\begin{aligned} \left\| \Psi_{|\varrho|}^{(i,\delta)}(c_k)_{k \in \mathbb{Z}^d} \right\|_{L_v^1} &\leq \sum_{k \in \mathbb{Z}^d} v_k^{(i,\delta)} |c_k| \cdot \left\| \frac{L_{\delta \cdot T_i^{-T} k} [\varrho \circ T_i^T]}{v_k^{(i,\delta)}} \right\|_{L_v^1} \\ &\leq \Omega_0^K \Omega_1 s_d \cdot |\det T_i|^{-1} \cdot \sum_{k \in \mathbb{Z}^d} v_k^{(i,\delta)} |c_k| \\ &= \Omega_0^K \Omega_1 s_d \cdot |\det T_i|^{-1} \cdot \|(c_k)_{k \in \mathbb{Z}^d}\|_{C_i^{(\delta)}} < \infty. \end{aligned}$$

This establishes boundedness of $\Psi_{|\varrho|}^{(i,\delta)}$ for $p = 1$.

As our next step, we first note

$$[M_Q(L_x f)](y) = \|\mathbf{1}_{y+Q} \cdot L_x f\|_{L^\infty} = \|(L_{-x} \mathbf{1}_{y+Q}) \cdot f\|_{L^\infty} = \|\mathbf{1}_{y-x+Q} \cdot f\|_{L^\infty} = (M_Q f)(y - x) = (L_x [M_Q f])(y)$$

for arbitrary measurable $f : \mathbb{R}^d \rightarrow \mathbb{C}$ and $Q \subset \mathbb{R}^d$. Hence,

$$\begin{aligned} g_i^{(\delta)}(x) &:= v(x) \cdot \left[M_{T_i^{-T}[-1,1]^d} \left(\frac{L_{\delta \cdot T_i^{-T} k} [\varrho \circ T_i^T]}{v_k^{(i,\delta)}} \right) \right](x) \\ &= \frac{v(x)}{v_k^{(i,\delta)}} \cdot \left[M_{T_i^{-T}[-1,1]^d} (\varrho \circ T_i^T) \right](x - \delta \cdot T_i^{-T} k) \\ (\text{eq. (5.9)}) &\leq \left[v_0 \cdot M_{T_i^{-T}[-1,1]^d} (\varrho \circ T_i^T) \right](x - \delta \cdot T_i^{-T} k) \\ (\text{Lemma 2.4}) &= \left(v_0 \cdot \left[(M_{[-1,1]^d} \varrho) \circ T_i^T \right] \right)(x - \delta \cdot T_i^{-T} k) \\ (\text{assumptions on } v_0 \text{ and eq. (1.11)}) &\leq \Omega_0^K \Omega_1 \cdot \left[(1 + |\bullet|)^K \cdot (M_{[-1,1]^d} \varrho) \right](T_i^T x - \delta \cdot k) \\ (\text{Lemma 2.3}) &\leq \Omega_0^K \Omega_1 \cdot \left(1 + 2\sqrt{d} \right)^{K_0} \cdot \left[(1 + |\bullet|)^{K-K_0} \right](T_i^T x - \delta k). \end{aligned}$$

But this implies for $p \in (0, 1)$ that

$$\begin{aligned} \left\| \frac{L_{\delta \cdot T_i^{-T} k} [\varrho \circ T_i^T]}{v_k^{(i, \delta)}} \right\|_{V_i} &= \|g_i^{(\delta)}\|_{L^p} \\ &\leq \Omega_0^K \Omega_1 \cdot (1 + 2\sqrt{d})^{K_0} \cdot \left\| [L_{\delta k} (1 + |\bullet|)^{K-K_0}] \circ T_i^T \right\|_{L^p} \\ &= \Omega_0^K \Omega_1 \cdot (1 + 2\sqrt{d})^{K_0} \cdot |\det T_i^T|^{-1/p} \cdot \|(1 + |\bullet|)^{K-K_0}\|_{L^p} \\ &\stackrel{(\text{eq. (1.9) and } K-K_0=-(\frac{d}{p}+1) \text{ since } p \in (0,1))}{\leq} \Omega_0^K \Omega_1 \cdot (1 + 2\sqrt{d})^{K_0} \left(\frac{s_d}{p}\right)^{1/p} \cdot |\det T_i|^{-1/p}. \end{aligned}$$

Now, we recall that for $p \in (0, 1)$, we have the p -triangle inequality $\|f + g\|_{L^p}^p \leq \|f\|_{L^p}^p + \|g\|_{L^p}^p$. By solidity and because of $M_Q(f + g) \leq M_Q f + M_Q g$, this also yields $\|f + g\|_{V_i}^p \leq \|f\|_{V_i}^p + \|g\|_{V_i}^p$, so that we get

$$\begin{aligned} \left\| \Psi_{|\varrho|}^{(i, \delta)}(c_k)_{k \in \mathbb{Z}^d} \right\|_{V_i}^p &\leq \sum_{k \in \mathbb{Z}^d} \left[v_k^{(i, \delta)} |c_k| \right]^p \cdot \left\| \frac{L_{\delta \cdot T_i^{-T} k} [\varrho \circ T_i^T]}{v_k^{(i, \delta)}} \right\|_{V_i}^p \\ &\leq \left[\Omega_0^K \Omega_1 \cdot (1 + 2\sqrt{d})^{K_0} \left(\frac{s_d}{p}\right)^{1/p} \cdot |\det T_i|^{-1/p} \right]^p \cdot \|(c_k)_{k \in \mathbb{Z}^d}\|_{C_i^{(\delta)}}^p, \end{aligned}$$

which yields the desired boundedness for $p \in (0, 1)$.

Next, we consider the case $p = \infty$. Here, we note because of $|\varrho(x)| = \varrho(x) = (1 + |x|)^{-K_0}$ that

$$\begin{aligned} v(x) \cdot \left| \left[\Psi_{|\varrho|}^{(i, \delta)}(c_k)_{k \in \mathbb{Z}^d} \right](x) \right| &= \left| \sum_{k \in \mathbb{Z}^d} c_k v_k^{(i, \delta)} \cdot \frac{v(x)}{v_k^{(i, \delta)}} \cdot (\varrho \circ T_i^T)(x - \delta \cdot T_i^{-T} k) \right| \\ &\stackrel{(\text{for } c=(c_k)_{k \in \mathbb{Z}^d} \text{ and since } p=\infty)}{\leq} \|c\|_{C_i^{(\delta)}} \cdot \sum_{k \in \mathbb{Z}^d} \left[\frac{v(x)}{v_k^{(i, \delta)}} \cdot (\varrho \circ T_i^T)(x - \delta \cdot T_i^{-T} k) \right] \\ &\stackrel{(\text{eq. (5.9) and assumption on } v_0)}{\leq} \Omega_1 \cdot \|c\|_{C_i^{(\delta)}} \cdot \sum_{k \in \mathbb{Z}^d} \left[(1 + |x - \delta \cdot T_i^{-T} k|)^K \cdot (\varrho \circ T_i^T)(x - \delta \cdot T_i^{-T} k) \right] \\ &\stackrel{(\text{eq. (1.11) and } \varrho(x)=(1+|x|)^{-K_0})}{\leq} \Omega_0^K \Omega_1 \cdot \|c\|_{C_i^{(\delta)}} \cdot \sum_{k \in \mathbb{Z}^d} \left[(1 + |T_i^T(x - \delta \cdot T_i^{-T} k)|)^{K-K_0} \right] \\ &= \Omega_0^K \Omega_1 \cdot \|c\|_{C_i^{(\delta)}} \cdot h(T_i^T x), \end{aligned}$$

where we introduced $h(y) := \sum_{k \in \mathbb{Z}^d} \left[(1 + |y - \delta \cdot k|)^{K-K_0} \right]$ for $y \in \mathbb{R}^d$ in the last step. Now, we recall $0 < \delta \leq 1$ and $K - K_0 \leq -(d+1) < 0$, so that

$$\begin{aligned} h(y) &\leq \sum_{k \in \mathbb{Z}^d} \left(1 + \left| \delta \cdot \left(\frac{y}{\delta} - k \right) \right| \right)^{-(d+1)} \\ &\leq \delta^{-(d+1)} \cdot \sum_{k \in \mathbb{Z}^d} \left(1 + \left| \frac{y}{\delta} - k \right| \right)^{-(d+1)} \\ &\stackrel{(\text{since } |x| \geq \|x\|_\infty)}{\leq} \delta^{-(d+1)} \cdot \sum_{k \in \mathbb{Z}^d} \left(1 + \left\| \frac{y}{\delta} - k \right\|_\infty \right)^{-(d+1)} =: \delta^{-(d+1)} \cdot \tilde{h}\left(\frac{y}{\delta}\right). \end{aligned}$$

Now, we note that \tilde{h} is \mathbb{Z}^d -periodic, so that $\|\tilde{h}\|_{\sup} = \|\tilde{h}\|_{\sup, [0,1]^d}$. But for arbitrary $x \in [0, 1]^d$, we have

$$1 + \|k\|_\infty \leq 1 + \|k - x\|_\infty + \|x\|_\infty \leq 2 + \|x - k\|_\infty \leq 2(1 + \|x - k\|_\infty)$$

and thus

$$\tilde{h}(x) = \sum_{k \in \mathbb{Z}^d} (1 + \|x - k\|_\infty)^{-(d+1)} \leq 2^{d+1} \cdot \sum_{k \in \mathbb{Z}^d} (1 + \|k\|_\infty)^{-(d+1)}.$$

Next, we use the layer-cake formula (cf. [29, Proposition (6.24)]) with the counting measure μ on \mathbb{Z}^d to estimate for $\theta(k) := (1 + \|k\|_\infty)^{-(d+1)}$ the series

$$\begin{aligned} \sum_{k \in \mathbb{Z}^d} (1 + \|k\|_\infty)^{-(d+1)} &= \int_{\mathbb{Z}^d} \theta(k) \, d\mu(k) = \int_0^\infty \mu(\{k \in \mathbb{Z}^d \mid \theta(k) > \lambda\}) \, d\lambda \\ &\left(\text{since } \theta(k) > \lambda \iff \|k\|_\infty < \lambda^{-\frac{1}{d+1}} - 1 \text{ can only hold for } \lambda < 1 \right) \leq \int_0^1 \mu\left(\left[-\lambda^{-\frac{1}{d+1}}, \lambda^{-\frac{1}{d+1}}\right]^d \cap \mathbb{Z}^d\right) \, d\lambda \\ &\leq \int_0^1 \mu\left(\left\{-\left\lfloor \lambda^{-\frac{1}{d+1}} \right\rfloor, \dots, \left\lfloor \lambda^{-\frac{1}{d+1}} \right\rfloor\right\}^d\right) \, d\lambda \\ &= \int_0^1 \left(1 + 2\left\lfloor \lambda^{-\frac{1}{d+1}} \right\rfloor\right)^d \, d\lambda \\ &\leq 3^d \cdot \int_0^1 \lambda^{-\frac{d}{d+1}} \, d\lambda = (d+1) \cdot 3^d \cdot \lambda^{\frac{1}{d+1}} \Big|_0^1 \\ &\quad (\text{since } 1+d \leq 2^d) \leq 6^d. \end{aligned} \tag{5.10}$$

Hence, we get $\tilde{h}(x) \leq 2^{d+1} \cdot 6^d \leq 2 \cdot 12^d$ for all $x \in [0, 1]^d$ and thus for all $x \in \mathbb{R}^d$, by \mathbb{Z}^d -periodicity. In view of the estimates from above, this entails for $c = (c_k)_{k \in \mathbb{Z}^d}$ that

$$v(x) \cdot \left| \left[\Psi_{|\varrho|}^{(i, \delta)}(c_k)_{k \in \mathbb{Z}^d} \right](x) \right| \leq \Omega_0^K \Omega_1 \cdot \|c\|_{C_i^{(\delta)}} \cdot h(T_i^T x) \leq \delta^{-(d+1)} \cdot 2 \cdot 12^d \cdot \Omega_0^K \Omega_1 \cdot \|c\|_{C_i^{(\delta)}} < \infty$$

for all $x \in \mathbb{R}^d$. In particular, the series defining $\Psi_{|\varrho|}^{(i, \delta)}$ converges pointwise absolutely in case of $p = \infty$. But since we have $\ell_{v^{(i, \delta)}}^p(\mathbb{Z}^d) \hookrightarrow \ell_{v^{(i, \delta)}}^\infty(\mathbb{Z}^d)$ for all $p \in (0, \infty]$, this implies absolute pointwise convergence for arbitrary $p \in (0, \infty]$.

Next, for $p \in [1, \infty]$, it is easy to see that the operator

$$\Lambda_{|\varrho|}^{(i, \delta)} : \ell^p(\mathbb{Z}^d) \rightarrow L^p(\mathbb{R}^d), (\zeta_k)_{k \in \mathbb{Z}^d} \mapsto v \cdot \sum_{k \in \mathbb{Z}^d} \frac{\zeta_k}{v_k^{(i, \delta)}} \cdot L_{\delta \cdot T_i^{-T} k} [|\varrho| \circ T_i^T]$$

is well-defined and bounded if and only if $\Psi_{|\varrho|}^{(i, \delta)}$ is, with $\left\| \Lambda_{|\varrho|}^{(i, \delta)} \right\| = \left\| \Psi_{|\varrho|}^{(i, \delta)} \right\|$. Hence, since we have already shown boundedness for $p \in \{1, \infty\}$, we can use complex interpolation (the Riesz-Thorin theorem, [29, Theorem (6.27)]) to derive for $p \in [1, \infty]$ that $\Psi_{|\varrho|}^{(i, \delta)}$ is bounded, with

$$\begin{aligned} \left\| \Psi_{|\varrho|}^{(i, \delta)} \right\| &\leq \left[\Omega_0^K \Omega_1 s_d \cdot |\det T_i|^{-1} \right]^{\frac{1}{p}} \cdot \left[\delta^{-(d+1)} \cdot 2 \cdot 12^d \cdot \Omega_0^K \Omega_1 \right]^{1-\frac{1}{p}} \\ &\quad (\text{since } s_d \leq 4^d) \leq \Omega_0^K \Omega_1 \cdot 12^{d+1} \cdot \delta^{-(1-\frac{1}{p})(d+1)} \cdot |\det T_i|^{-\frac{1}{p}}. \end{aligned}$$

Finally, for $g \in L_{v_0}^1(\mathbb{R}^d)$ and $c = (c_k)_{k \in \mathbb{Z}^d} \in C_i^{(\delta)} = \ell_{v^{(i, \delta)}}^p(\mathbb{Z}^d) \hookrightarrow \ell_{v^{(i, \delta)}}^\infty(\mathbb{Z}^d)$, our previous considerations, in combination with $v(x) = v(x-y+y) \leq v(x-y)v_0(y)$, show for arbitrary measurable $\varrho : \mathbb{R}^d \rightarrow \mathbb{C}$ with $\|\varrho\|_{K_0} < \infty$ that

$$\begin{aligned} \int_{\mathbb{R}^d} |g(y)| \cdot \sum_{k \in \mathbb{Z}^d} |c_k| \cdot \left| \left(L_{\delta \cdot T_i^{-T} k} [\varrho \circ T_i^T] \right)(x-y) \right| \, dy &\leq \frac{1}{v(x)} \cdot \int_{\mathbb{R}^d} |v_0(y) \cdot g(y)| \cdot v(x-y) \cdot \left[\Psi_{|\varrho|}^{(i, \delta)}(|c_k|)_{k \in \mathbb{Z}^d} \right](x-y) \, dy \\ &\leq \frac{1}{v(x)} \cdot \left\| \Psi_{|\varrho|}^{(i, \delta)}(|c_k|)_{k \in \mathbb{Z}^d} \right\|_{L_{v_0}^\infty} \cdot \|g\|_{L_{v_0}^1} < \infty \end{aligned}$$

for all $x \in \mathbb{R}^d$, so that the interchange of summation and integration in

$$\begin{aligned} \left[g * \Psi_{|\varrho|}^{(i, \delta)}(c_k)_{k \in \mathbb{Z}^d} \right](x) &= \int_{\mathbb{R}^d} g(y) \cdot \left(\sum_{k \in \mathbb{Z}^d} c_k \cdot L_{\delta \cdot T_i^{-T} k} [\varrho \circ T_i^T] \right)(x-y) \, dy \\ &= \sum_{k \in \mathbb{Z}^d} c_k \cdot \int_{\mathbb{R}^d} g(y) \cdot \left(L_{\delta \cdot T_i^{-T} k} [\varrho \circ T_i^T] \right)(x-y) \, dy \\ &= \sum_{k \in \mathbb{Z}^d} c_k \cdot \left(g * L_{\delta \cdot T_i^{-T} k} [\varrho \circ T_i^T] \right)(x) \end{aligned}$$

is justified by the dominated convergence theorem. \square

As a further ingredient, we will need the following “sampling theorem” for bandlimited functions. A very similar statement already appears in [73, Proposition in §1.3.3], so no originality at all is claimed. Note, however, that in [73], it is assumed that $\varphi \in \mathcal{S}(\mathbb{R}^d)$ instead of $\varphi \in \mathcal{S}'(\mathbb{R}^d)$ and furthermore, the statement in [73] is restricted to the *unweighted* case.

Lemma 5.4. *For each $i \in I$, $R > 0$ and $p \in (0, \infty]$, as well as*

$$C := 2^{\max\{1, \frac{1}{p}\}} \cdot \Omega_0^{3K} \Omega_1^3 \cdot (1 + \sqrt{d})^K \cdot \left(23040 \cdot d^{3/2} \cdot \left(K + 1 + \frac{d+1}{\min\{1, p\}} \right) \right)^{K+2+\frac{d+1}{\min\{1, p\}}} \cdot (1+R)^{1+\frac{d}{\min\{1, p\}}}$$

we have

$$\left\| [\varphi(\delta \cdot T_i^{-T} k)]_{k \in \mathbb{Z}^d} \right\|_{C_i^{(\delta)}} \leq C \cdot \delta^{-d/p} \cdot |\det T_i|^{1/p} \cdot \|\varphi\|_{L_v^p}$$

for all $\delta \in (0, 1]$ and all $\varphi \in \mathcal{S}'(\mathbb{R}^d)$ with $\text{supp } \widehat{\varphi} \subset T_i[-R, R]^d + \xi_0$, for arbitrary $\xi_0 \in \mathbb{R}^d$. ◀

Proof. Clearly, we can assume without loss of generality that $\|\varphi\|_{L_v^p} < \infty$.

Let us first consider the case $p = \infty$. Since we have $v(x) \cdot |\varphi(x)| \leq \|\varphi\|_{L_v^\infty}$ for almost all $x \in \mathbb{R}^d$, and thus for a dense subset of \mathbb{R}^d , there is for arbitrary $k \in \mathbb{Z}^d$ a sequence $(x_n)_{n \in \mathbb{N}}$ satisfying $x_n \xrightarrow{n \rightarrow \infty} \delta \cdot T_i^{-T} k$ as well as $v(x_n) \cdot |\varphi(x_n)| \leq \|\varphi\|_{L_v^\infty}$. But since φ is given by (integration against) a continuous function by the Paley-Wiener theorem, this implies

$$\begin{aligned} v(\delta \cdot T_i^{-T} k) \cdot |\varphi(\delta \cdot T_i^{-T} k)| &= \lim_{n \rightarrow \infty} v(\delta \cdot T_i^{-T} k) \cdot |\varphi(x_n)| \\ &\leq \liminf_{n \rightarrow \infty} v(x_n + \delta \cdot T_i^{-T} k - x_n) \cdot |\varphi(x_n)| \\ &\leq \liminf_{n \rightarrow \infty} v(x_n) \cdot |\varphi(x_n)| \cdot v_0(\delta \cdot T_i^{-T} k - x_n) \\ &\leq \Omega_1 \cdot \|\varphi\|_{L_v^\infty} \cdot \liminf_{n \rightarrow \infty} (1 + |\delta \cdot T_i^{-T} k - x_n|)^K \\ &= \Omega_1 \cdot \|\varphi\|_{L_v^\infty} < \infty. \end{aligned}$$

Since $C \geq \Omega_1$, since $\delta^{-d/p} \cdot |\det T_i|^{1/p} = 1$ for $p = \infty$ and since $k \in \mathbb{Z}^d$ was arbitrary, this establishes the claim for $p = \infty$. Hence, we can assume $p \in (0, \infty)$ in what follows.

Let $C > 0$ as in the statement of the theorem and $C_1 := 2^{-\max\{1, \frac{1}{p}\}} \cdot \left[\Omega_0^K \Omega_1 \cdot (1 + \sqrt{d})^K \right]^{-1} \cdot C$. Let $\varrho := M_{-\xi_0} \varphi$ and note $|\varrho| = |\varphi|$. Now, Theorem 2.18 shows

$$\left\| \text{osc}_{\delta \cdot T_i^{-T}[-1, 1]^d} \varrho \right\|_{W_{T_i^{-T}[-1, 1]^d}(L_v^p)} = \left\| \text{osc}_{\delta \cdot T_i^{-T}[-1, 1]^d} [M_{-\xi_0} \varphi] \right\|_{W_{T_i^{-T}[-1, 1]^d}(L_v^p)} \leq C_1 \cdot \delta \cdot \|\varphi\|_{L_v^p}$$

for all $\delta \in (0, 1]$ and φ as in the statement of the lemma.

Now, notice for arbitrary $k \in \mathbb{Z}^d$ and $x \in \delta T_i^{-T}(k + [0, 1]^d)$ that $\delta T_i^{-T} k \in x - \delta T_i^{-T}[0, 1]^d \subset x + \delta T_i^{-T}[-1, 1]^d$ and hence

$$\begin{aligned} |\varphi(\delta \cdot T_i^{-T} k)| &= |\varrho(\delta \cdot T_i^{-T} k)| \leq |\varrho(x)| + |\varrho(x) - \varrho(\delta \cdot T_i^{-T} k)| \\ &\leq |\varphi(x)| + \left(\text{osc}_{\delta \cdot T_i^{-T}[-1, 1]^d} \varrho \right)(x). \end{aligned}$$

By multiplying this estimate with $\mathbb{1}_{\delta T_i^{-T}(k + [0, 1]^d)}(x)$, summing over $k \in \mathbb{Z}^d$ and using $\mathbb{R}^d = \bigsqcup_{k \in \mathbb{Z}^d} \delta T_i^{-T}(k + [0, 1]^d)$, we obtain

$$\sum_{k \in \mathbb{Z}^d} \left(\mathbb{1}_{\delta T_i^{-T}(k + [0, 1]^d)}(x) \cdot |\varphi(\delta T_i^{-T} k)| \right) \leq |\varphi(x)| + \left(\text{osc}_{\delta \cdot T_i^{-T}[-1, 1]^d} \varrho \right)(x) \quad \forall x \in \mathbb{R}^d.$$

By solidity of $L_v^p(\mathbb{R}^d)$, we conclude

$$\begin{aligned}
& \left\| \sum_{k \in \mathbb{Z}^d} \left(\mathbb{1}_{\delta T_i^{-T}(k+[0,1]^d)} \cdot |\varphi(\delta T_i^{-T}k)| \right) \right\|_{L_v^p} \\
& \leq \left\| |\varphi| + \text{osc}_{\delta \cdot T_i^{-T}[-1,1]^d} \varphi \right\|_{L_v^p} \\
& \left(C_2 := 2^{\max\{0, \frac{1}{p}-1\}} \text{ is triangle const. for } L_v^p \right) \leq C_2 \cdot \left(\|\varphi\|_{L_v^p} + \left\| \text{osc}_{\delta \cdot T_i^{-T}[-1,1]^d} \varphi \right\|_{L_v^p} \right) \\
& \leq C_2 \cdot (\|\varphi\|_{L_v^p} + C_1 \cdot \delta \cdot \|\varphi\|_{L_v^p}) \\
& \quad (\text{since } \delta \leq 1) \leq C_2 (1 + C_1) \cdot \|\varphi\|_{L_v^p} \\
& \quad (\text{since } C_1 \geq 1) \leq 2^{\max\{1, \frac{1}{p}\}} \cdot C_1 \cdot \|\varphi\|_{L_v^p}.
\end{aligned}$$

Finally, we note for $x \in \delta \cdot T_i^{-T}(k+[0,1]^d)$, i.e., for $x = \delta T_i^{-T}k + \delta T_i^{-T}q$ with $q \in [0,1]^d$ that

$$\begin{aligned}
v_k^{(i,\delta)} &= v(\delta \cdot T_i^{-T}k) = v(x - \delta T_i^{-T}q) \leq v(x) \cdot v_0(\delta \cdot T_i^{-T}q) \\
& \quad (\text{assump. on } v_0 \text{ and eq. (1.11)}) \leq \Omega_1 \cdot v(x) \cdot (1 + |\delta \cdot T_i^{-T}q|)^K \leq \Omega_0^K \Omega_1 \cdot v(x) \cdot (1 + |\delta \cdot q|)^K \\
& \quad (\text{since } \delta \leq 1) \leq \Omega_0^K \Omega_1 \cdot (1 + \sqrt{d})^K \cdot v(x).
\end{aligned}$$

Using this estimate and the pairwise disjointness of $(\delta \cdot T_i^{-T}(k+[0,1]^d))_{k \in \mathbb{Z}^d}$, we conclude

$$\begin{aligned}
& \left\| \sum_{k \in \mathbb{Z}^d} \left(\mathbb{1}_{\delta T_i^{-T}(k+[0,1]^d)} \cdot |\varphi(\delta T_i^{-T}k)| \right) \right\|_{L_v^p} = \left(\sum_{k \in \mathbb{Z}^d} |\varphi(\delta \cdot T_i^{-T}k)|^p \int_{\delta T_i^{-T}(k+[0,1]^d)} [v(x)]^p dx \right)^{1/p} \\
& \geq \left[\Omega_0^K \Omega_1 \cdot (1 + \sqrt{d})^K \right]^{-1} \cdot \left(\sum_{k \in \mathbb{Z}^d} \left[v_k^{(i,\delta)} \cdot |\varphi(\delta T_i^{-T}k)| \right]^p \cdot \lambda_d(\delta T_i^{-T}(k+[0,1]^d)) \right)^{\frac{1}{p}} \\
& = \left[\Omega_0^K \Omega_1 \cdot (1 + \sqrt{d})^K \right]^{-1} \cdot \delta^{d/p} \cdot |\det T_i|^{-1/p} \cdot \left\| [\varphi(\delta \cdot T_i^{-T}k)]_{k \in \mathbb{Z}^d} \right\|_{C_i^{(\delta)}}.
\end{aligned}$$

Putting everything together, we conclude

$$\begin{aligned}
& \left\| [\varphi(\delta \cdot T_i^{-T}k)]_{k \in \mathbb{Z}^d} \right\|_{C_i^{(\delta)}} \leq \delta^{-\frac{d}{p}} \cdot |\det T_i|^{\frac{1}{p}} \cdot \Omega_0^K \Omega_1 \cdot (1 + \sqrt{d})^K \cdot \left\| \sum_{k \in \mathbb{Z}^d} \left(\mathbb{1}_{\delta T_i^{-T}(k+[0,1]^d)} \cdot |\varphi(\delta T_i^{-T}k)| \right) \right\|_{L_v^p} \\
& \leq \delta^{-\frac{d}{p}} \cdot |\det T_i|^{\frac{1}{p}} \cdot \Omega_0^K \Omega_1 \cdot (1 + \sqrt{d})^K \cdot 2^{\max\{1, \frac{1}{p}\}} \cdot C_1 \cdot \|\varphi\|_{L_v^p} \\
& = \delta^{-\frac{d}{p}} \cdot |\det T_i|^{\frac{1}{p}} \cdot C \cdot \|\varphi\|_{L_v^p} < \infty,
\end{aligned}$$

as desired. \square

In the proof of Theorem 5.6, we will employ a Neumann series argument for an operator defined on $\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$. For this to be justified, we need to know that this space is a Quasi-Banach space, i.e., complete. For $v \equiv 1$, this was already shown in [77, Theorem 3.21], but for $v \not\equiv 1$ and general $p, q \in (0, \infty]$, it seems that the following lemma is a novel (though not too surprising) result:

Lemma 5.5. *The decomposition space $\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$ is a Quasi-Banach space.* \blacktriangleleft

Proof. Verifying the quasi-norm properties of $\|\bullet\|_{\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)}$ is relatively straightforward (and essentially identical to the verification in [77, Theorem 3.21]), so we only prove completeness.

Instead of verifying completeness directly, we use a slightly more abstract approach, employing other results from the paper. The main new ingredient that we need to provide is boundedness of

$$\text{Ana}_* : \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q) \rightarrow V = \ell_w^q([V_i]_{i \in I}), f \mapsto \left[\mathcal{F}^{-1}(\varphi_i^* \hat{f}) \right]_{i \in I}.$$

To this end, let $f \in \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$ be arbitrary and define $c_i := \left\| \mathcal{F}^{-1}(\varphi_i \hat{f}) \right\|_{L_v^p}$ for $i \in I$. Note $c = (c_i)_{i \in I} \in \ell_w^q(I)$ and $\|c\|_{\ell_w^q} = \|f\|_{\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)}$. Recall that the clustering map $\Gamma_{\mathcal{Q}} : \ell_w^q(I) \rightarrow \ell_w^q(I)$ with $\Gamma_{\mathcal{Q}}(e_i)_{i \in I} = (e_i^*)_{i \in I}$ and $e_i^* = \sum_{\ell \in i^*} e_\ell$ is bounded.

Now, we distinguish the cases $p \in [1, \infty]$ and $p \in (0, 1)$. In case of $p \in [1, \infty]$, the triangle inequality for $L_v^p(\mathbb{R}^d)$ yields because of $V_i = L_v^p(\mathbb{R}^d)$ for all $i \in I$ that

$$\begin{aligned} \|\text{Ana}_* f\|_V &= \left\| \left(\left\| \mathcal{F}^{-1}(\varphi_i^* \cdot \hat{f}) \right\|_{V_i} \right)_{i \in I} \right\|_{\ell_w^q} \leq \left\| \left(\sum_{\ell \in i^*} \left\| \mathcal{F}^{-1}(\varphi_\ell \cdot \hat{f}) \right\|_{L_v^p} \right)_{i \in I} \right\|_{\ell_w^q} \\ &= \|\Gamma_{\mathcal{Q}} c\|_{\ell_w^q} \leq \|\Gamma_{\mathcal{Q}}\| \cdot \|c\|_{\ell_w^q} = \|\Gamma_{\mathcal{Q}}\| \cdot \|f\|_{\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)} < \infty. \end{aligned}$$

Now, we consider the case $p \in (0, 1)$. We observe that [77, Lemma 2.7] yields some $R = R(R_{\mathcal{Q}}, C_{\mathcal{Q}}) > 0$ satisfying $\overline{Q_i^*} \subset T_i \overline{B_R(0)} + b_i \subset T_i [-R, R]^d + b_i$ for all $i \in I$. Because of $\text{supp}(\varphi_i^* \hat{f}) \subset \overline{Q_i^*}$, Theorem 2.17 thus yields a constant $C_1 = C_1(d, p, R, \Omega_0, \Omega_1, K) > 0$ such that

$$\left\| \mathcal{F}^{-1}(\varphi_i^* \cdot \hat{f}) \right\|_{V_i} = \left\| \mathcal{F}^{-1}(\varphi_i^* \cdot \hat{f}) \right\|_{W_{T_i^{-T}[-1,1]^d}(L_v^p)} \leq C_1 \cdot \left\| \mathcal{F}^{-1}(\varphi_i^* \cdot \hat{f}) \right\|_{L_v^p} \quad \forall i \in I.$$

Next, since $L_v^p(\mathbb{R}^d)$ is a Quasi-Banach space and since we have the uniform estimate $|i^*| \leq N_{\mathcal{Q}}$ for all $i \in I$, there is a constant $C_2 = C_2(N_{\mathcal{Q}}, p) > 0$ satisfying

$$\left\| \mathcal{F}^{-1}(\varphi_i^* \cdot \hat{f}) \right\|_{L_v^p} \leq C_2 \cdot \sum_{\ell \in i^*} \left\| \mathcal{F}^{-1}(\varphi_\ell \cdot \hat{f}) \right\|_{L_v^p} = C_2 \cdot (\Gamma_{\mathcal{Q}} c)_i \quad \forall i \in I.$$

All in all, this entails by solidity of $\ell_w^q(I)$ that

$$\|\text{Ana}_* f\|_V = \left\| \left(\left\| \mathcal{F}^{-1}(\varphi_i^* \cdot \hat{f}) \right\|_{V_i} \right)_{i \in I} \right\|_{\ell_w^q} \leq C_1 C_2 \cdot \|\Gamma_{\mathcal{Q}} c\|_{\ell_w^q} \leq C_1 C_2 \|\Gamma_{\mathcal{Q}}\| \cdot \|f\|_{\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)} < \infty,$$

as above. In summary, Ana_* is well-defined and bounded for all $p \in (0, \infty]$.

Now, using the map $\text{Synth}_{\mathcal{D}}$ from Lemma 3.9, we have because of $\varphi_i^* \varphi_i = \varphi_i$ that

$$(\text{Synth}_{\mathcal{D}} \circ \text{Ana}_*) f = \sum_{i \in I} \mathcal{F}^{-1}(\varphi_i \cdot \mathcal{F}[\text{Ana}_* f]_i) = \sum_{i \in I} \mathcal{F}^{-1}(\varphi_i \varphi_i^* \cdot \hat{f}) = \sum_{i \in I} \mathcal{F}^{-1}(\varphi_i \cdot \hat{f}) = f$$

for all $f \in \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$. Finally, recall from (the remark after) Lemma 4.5 that $V = \ell_w^q([V_i]_{i \in I})$ is complete.

Now, let $(f_n)_{n \in \mathbb{N}}$ be Cauchy in $\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$. Since Ana_* is bounded, the sequence $(g_n)_{n \in \mathbb{N}}$ with $g_n := \text{Ana}_* f_n$ is Cauchy in V . Hence, $g_n \rightarrow g$ for some $g \in V$. Define $f := \text{Synth}_{\mathcal{D}} g \in \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$ and observe

$$\|f_n - f\|_{\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)} = \|\text{Synth}_{\mathcal{D}} \text{Ana}_* f_n - \text{Synth}_{\mathcal{D}} g\|_{\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)} \leq \|\text{Synth}_{\mathcal{D}}\| \cdot \|g_n - g\|_V \xrightarrow{n \rightarrow \infty} 0. \quad \square$$

Using all of the technical lemmata collected in this section, we can finally prove that the family $\left(L_{\delta \cdot T_j^{-T} k} \gamma^{[j]} \right)_{k \in \mathbb{Z}^d, j \in I}$ yields an atomic decomposition of $\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$, if $\delta > 0$ is chosen small enough.

Theorem 5.6. Assume that the families $\Gamma = (\gamma_i)_{i \in I}$ and $\Gamma_\ell = (\gamma_{i,\ell})_{i \in I}$ with $\ell \in \{1, 2\}$ satisfy Assumption 5.1 and that $\Gamma = (\gamma_i)_{i \in I}$ satisfies Assumption 3.6. Define $\delta_0 > 0$ by

$$\delta_0^{-1} := \begin{cases} \frac{2s_d}{\sqrt{d}} \cdot (2^{17} \cdot d^2 \cdot (K+2+d))^{K+d+3} \cdot (1+R_{\mathcal{Q}})^{d+1} \cdot \Omega_0^{4K} \Omega_1^{4K} \Omega_2^{(p,K)} \Omega_4^{(p,K)} \cdot \|\vec{C}\|, & \text{if } p \geq 1, \\ \frac{(2^{14}/d^{\frac{3}{2}})^{\frac{d}{p}}}{2^{45} \cdot d^{17}} \cdot \left(\frac{s_d}{p} \right)^{\frac{1}{p}} \left(2^{68} \cdot d^{14} \cdot \left(K+1+\frac{d+1}{p} \right)^3 \right)^{K+2+\frac{d+1}{p}} \cdot (1+R_{\mathcal{Q}})^{1+\frac{3d}{p}} \cdot \Omega_0^{16K} \Omega_1^{16K} \Omega_2^{(p,K)} \Omega_4^{(p,K)} \cdot \|\vec{C}\|^{\frac{1}{p}}, & \text{if } p < 1. \end{cases}$$

Then, for each $0 < \delta \leq \min\{1, \delta_0\}$, the family $\left(L_{\delta \cdot T_j^{-T} k} \gamma^{[j]} \right)_{j \in I, k \in \mathbb{Z}^d}$ forms an **atomic decomposition** of $\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$. Precisely, this means the following:

(1) The **synthesis map**

$$\begin{aligned} S^{(\delta)} : \ell_{\left(|\det T_j|^{\frac{1}{2} - \frac{1}{p}} w_j \right)_{j \in I}}^q &\left([C_j^{(\delta)}]_{j \in I} \right) \rightarrow \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q), \\ (c_k^{(j)})_{j \in I, k \in \mathbb{Z}^d} &\mapsto \sum_{j \in I} \sum_{k \in \mathbb{Z}^d} \left(|\det T_j|^{-\frac{1}{2}} c_k^{(j)} \cdot L_{\delta \cdot T_j^{-T} k} \gamma^{(j)} \right) = \sum_{j \in I} \sum_{k \in \mathbb{Z}^d} \left(c_k^{(j)} \cdot L_{\delta \cdot T_j^{-T} k} \gamma^{[j]} \right) \end{aligned}$$

is well-defined and bounded for each $\delta \in (0, 1]$.

(2) For $0 < \delta \leq \min \{1, \delta_0\}$, there is a bounded linear **coefficient map**

$$C^{(\delta)} : \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q) \rightarrow \ell^q \left(|\det T_j|^{\frac{1}{2} - \frac{1}{p} w_j} \right)_{j \in I} \left([C_j^{(\delta)}]_{j \in I} \right)$$

satisfying $S^{(\delta)} \circ C^{(\delta)} = \text{id}_{\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)}$. ◀

Remark. As the proof shows, convergence of the series $\sum_{j \in I} \sum_{k \in \mathbb{Z}^d} \left(|\det T_j|^{-\frac{1}{2}} c_k^{(j)} \cdot L_{\delta \cdot T_j^{-T} k} \gamma^{(j)} \right)$ has to be understood as follows: Each of the series

$$\sum_{k \in \mathbb{Z}^d} \left(|\det T_j|^{-\frac{1}{2}} c_k^{(j)} \cdot L_{\delta \cdot T_j^{-T} k} \gamma^{(j)} \right)$$

converges pointwise absolutely to a function $g_j \in V_j \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$ and the series $\sum_{j \in I} g_j$ converges in the weak-* sense in $Z'(\mathcal{O})$, i.e., for every $\phi \in Z(\mathcal{O})$, the series $\sum_{j \in I} \langle g_j, \phi \rangle_{\mathcal{S}', \mathcal{S}}$ converges absolutely and the functional $\phi \mapsto \sum_{j \in I} \langle g_j, \phi \rangle_{\mathcal{S}', \mathcal{S}}$ is continuous on $Z(\mathcal{O})$.

Furthermore, the proof shows that the definition of $C^{(\delta)}$ is independent of the precise choice of p, q, v, w , as long as $\delta > 0$ is chosen small enough that $C^{(\delta)}$ is defined at all. In fact, the proof shows that $C^{(\delta)} = D^{(\delta)} \cdot (T^{(\delta)})^{-1}$ where $T^{(\delta)} = S^{(\delta)} \circ D^{(\delta)} : \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q) \rightarrow \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$ is invertible (using a Neumann series) for $0 < \delta \leq \min \{1, \delta_0\}$, with

$$\begin{aligned} D^{(\delta)} : \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q) &\rightarrow \ell^q \left(|\det T_j|^{\frac{1}{2} - \frac{1}{p} w_j} \right)_{j \in I} \left([C_j^{(\delta)}]_{j \in I} \right), \\ f &\mapsto \left[\left(\delta^d \cdot |\det T_j|^{-1/2} \cdot \left[\mathcal{F}^{-1}(\theta_j \varphi_j \cdot \widehat{f}) \right] (\delta \cdot T_j^{-T} k) \right)_{k \in \mathbb{Z}^d} \right]_{j \in I}, \end{aligned}$$

where θ_j for $j \in I$ is defined as in Assumption 3.6. ◆

Proof. We first study for arbitrary $j \in I$ and $\delta \in (0, 1]$ boundedness (and well-definedness) of the map

$$S_{\Gamma_2}^{(\delta, j)} : C_j^{(\delta)} \rightarrow V_j, (c_k)_{k \in \mathbb{Z}^d} \mapsto |\det T_j|^{-1/2} \cdot \sum_{k \in \mathbb{Z}^d} c_k \cdot L_{\delta \cdot T_j^{-T} k} \gamma_2^{(j)}.$$

Recall $\gamma_2^{(j)} = |\det T_j| \cdot M_{b_j} [\gamma_{j,2} \circ T_j^T]$, so that

$$\begin{aligned} \left[S_{\Gamma_2}^{(\delta, j)} (c_k)_{k \in \mathbb{Z}^d} \right] (x) &= |\det T_j|^{-1/2} \cdot \sum_{k \in \mathbb{Z}^d} c_k \cdot \gamma_2^{(j)} (x - \delta \cdot T_j^{-T} k) \\ &= |\det T_j|^{1/2} \cdot \sum_{k \in \mathbb{Z}^d} e^{2\pi i \langle b_j, x - \delta \cdot T_j^{-T} k \rangle} \cdot c_k \cdot \gamma_{j,2} (T_j^T x - \delta \cdot k) \\ &= |\det T_j|^{1/2} \cdot e^{2\pi i \langle b_j, x \rangle} \cdot \left(\sum_{k \in \mathbb{Z}^d} e^{-2\pi i \langle b_j, \delta \cdot T_j^{-T} k \rangle} c_k \cdot L_{\delta \cdot k} \gamma_{j,2} \right) (T_j^T x) \\ &\quad \left(\text{with } \Psi_{\gamma_{j,2}}^{(j, \delta)} \text{ as in Lemma 5.3} \right) = |\det T_j|^{1/2} \cdot e^{2\pi i \langle b_j, x \rangle} \cdot \Psi_{\gamma_{j,2}}^{(j, \delta)} \left[\left(e^{-2\pi i \langle b_j, \delta \cdot T_j^{-T} k \rangle} c_k \right)_{k \in \mathbb{Z}^d} \right]. \end{aligned} \quad (5.11)$$

Thus, in terms of the isometric isomorphism $m_j^{(\delta)} : C_j^{(\delta)} \rightarrow C_j^{(\delta)}, (c_k)_{k \in \mathbb{Z}^d} \mapsto \left(e^{-2\pi i \langle b_j, \delta \cdot T_j^{-T} k \rangle} \cdot c_k \right)_{k \in \mathbb{Z}^d}$ and of the map $\Psi_{\gamma_{j,2}}^{(j, \delta)}$ defined in Lemma 5.3, the preceding calculations show

$$S_{\Gamma_2}^{(\delta, j)} c = |\det T_j|^{1/2} \cdot M_{b_j} \left[\Psi_{\gamma_{j,2}}^{(j, \delta)} \left(m_j^{(\delta)} c \right) \right] \quad \forall c \in C_j^{(\delta)}.$$

As a consequence of the solidity of V_j and of Lemma 5.3 (which is applicable, since $\|\gamma_{j,2}\|_{K_0} \leq \Omega_4^{(p, K)} < \infty$ for all $j \in I$, cf. Assumption 5.1), we thus get

$$\left\| S_{\Gamma_2}^{(\delta, j)} \right\| \leq C_{K, \delta, d, p, \Gamma_2} \cdot |\det T_j|^{\frac{1}{2} - \frac{1}{p}} < \infty \quad \forall j \in I \quad (5.12)$$

for a suitable constant $C_{K, \delta, d, p, \Gamma_2} > 0$ which is independent of $j \in I$. In particular, each map $S_{\Gamma_2}^{(\delta, j)}$ is well-defined with pointwise absolute convergence of the defining series.

Now, we can establish boundedness of the synthesis map $S^{(\delta)}$ as follows: In view of equation (5.12), it follows that

$$\bigotimes_{j \in I} S_{\Gamma_2}^{(\delta, j)} : \ell_w^q \left(\left([\det T_j]^{\frac{1}{2} - \frac{1}{p}} \cdot w_j \right)_{j \in I} \right) \rightarrow \ell_w^q \left([V_j]_{j \in I} \right), (c_k^{(j)})_{j \in I, k \in \mathbb{Z}^d} \mapsto \left[S_{\Gamma_2}^{(\delta, j)} (c_k^{(j)})_{k \in \mathbb{Z}^d} \right]_{j \in I}$$

is well-defined and bounded, with $\left\| \bigotimes_{j \in I} S_{\Gamma_2}^{(\delta, j)} \right\| \leq C_{K, \delta, d, p, \Gamma_2}$. Furthermore, using $\gamma_j = \gamma_{j,1} * \gamma_{j,2}$, it follows easily that $L_x \gamma^{(j)} = \gamma_1^{(j)} * L_x \gamma_2^{(j)}$ for arbitrary $x \in \mathbb{R}^d$ and $j \in I$, from which it follows (with Synth_{Γ_1} as in Lemma 5.2) that

$$\begin{aligned} \left[\text{Synth}_{\Gamma_1} \circ \bigotimes_{j \in I} S_{\Gamma_2}^{(\delta, j)} \right] (c_k^{(j)})_{j \in I, k \in \mathbb{Z}^d} &= \sum_{j \in I} \left[\gamma_1^{(j)} * S_{\Gamma_2}^{(\delta, j)} (c_k^{(j)})_{k \in \mathbb{Z}^d} \right] \\ &= \sum_{j \in I} \left[|\det T_j|^{-1/2} \cdot \gamma_1^{(j)} * \sum_{k \in \mathbb{Z}^d} c_k^{(j)} \cdot L_{\delta \cdot T_j^{-T} k} \gamma_2^{(j)} \right] \\ &\stackrel{(\gamma_1^{(j)} \in L_{v_0}^1(\mathbb{R}^d) \text{ and (proof of) Lemma 5.3})}{=} \sum_{j \in I} \left[|\det T_j|^{-1/2} \cdot \sum_{k \in \mathbb{Z}^d} c_k^{(j)} \cdot L_{\delta \cdot T_j^{-T} k} \left(\gamma_1^{(j)} * \gamma_2^{(j)} \right) \right] \\ &= \sum_{j \in I} \left[\sum_{k \in \mathbb{Z}^d} |\det T_j|^{-1/2} \cdot c_k^{(j)} \cdot L_{\delta \cdot T_j^{-T} k} \gamma^{(j)} \right] \\ &= S^{(\delta)} (c_k^{(j)})_{j \in I, k \in \mathbb{Z}^d}. \end{aligned} \tag{5.13}$$

This shows for arbitrary $\delta \in (0, 1]$ that $S^{(\delta)} = \text{Synth}_{\Gamma_1} \circ \bigotimes_{j \in I} S_{\Gamma_2}^{(\delta, j)}$ is bounded, as a composition of bounded maps.

Finally, we prove that convergence of the series defining $S^{(\delta)} (c_k^{(j)})_{j \in I, k \in \mathbb{Z}^d}$ occurs in the sense described in the remark following the theorem: First of all, we get exactly as in equation (5.11) (but with γ_j instead of $\gamma_{j,2}$) that

$$g_j := \sum_{k \in \mathbb{Z}^d} \left(|\det T_j|^{-\frac{1}{2}} c_k^{(j)} \cdot L_{\delta \cdot T_j^{-T} k} \gamma^{(j)} \right) = |\det T_j|^{1/2} \cdot M_{b_j} \left[\Psi_{\gamma_j}^{(j, \delta)} \left(m_j^{(\delta)} (c_k^{(j)})_{k \in \mathbb{Z}^d} \right) \right].$$

Here, it is worth mentioning that the conditions in Assumption 5.1 imply that Lemma 5.3 is applicable with $\varrho = \gamma_{j,2}$, as well as with $\varrho = \gamma_j$. Hence, that lemma shows that the series defining g_j converges pointwise absolutely and that $g_j \in V_j \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$, where the last embedding is justified by Lemma 3.3.

Finally, Lemma 5.2 shows that for $(h_j)_{j \in I} := \left(\bigotimes_{j \in I} S_{\Gamma_2}^{(\delta, j)} \right) (c_k^{(j)})_{j \in I, k \in \mathbb{Z}^d} \in \ell_w^q \left([V_j]_{j \in I} \right)$ and each $\phi \in Z(\mathcal{O})$, the series

$$\begin{aligned} \left\langle \text{Synth}_{\Gamma_1} (h_j)_{j \in I}, \phi \right\rangle_{Z'(\mathcal{O}), Z(\mathcal{O})} &= \sum_{j \in I} \left\langle \gamma_1^{(j)} * h_j, \phi \right\rangle_{\mathcal{S}', \mathcal{S}} \\ &= \sum_{j \in I} \left\langle \gamma_1^{(j)} * \left[|\det T_j|^{-1/2} \cdot \sum_{k \in \mathbb{Z}^d} c_k^{(j)} L_{\delta \cdot T_j^{-T} k} \gamma_2^{(j)} \right], \phi \right\rangle_{\mathcal{S}', \mathcal{S}} \\ &\stackrel{(\text{cf. eq. (5.13)})}{=} \sum_{j \in I} \left\langle |\det T_j|^{-1/2} \cdot \sum_{k \in \mathbb{Z}^d} c_k^{(j)} L_{\delta \cdot T_j^{-T} k} \gamma^{(j)}, \phi \right\rangle_{\mathcal{S}', \mathcal{S}} \\ &= \sum_{j \in I} \langle g_j, \phi \rangle_{\mathcal{S}', \mathcal{S}} \end{aligned}$$

converges absolutely and defines a continuous functional on $Z(\mathcal{O})$.

Next, we want to show existence of the coefficient map $C^{(\delta)}$, for $0 < \delta \leq \min \{1, \delta_0\}$. To this end, first note that Theorem 2.17 shows for

$$C_1 := \begin{cases} 1, & \text{if } p \geq 1, \\ 2^{d(1+\frac{d}{p})} s_d^{\frac{1}{p}} \left(192 \cdot d^{3/2} \cdot \left\lceil K + \frac{d+1}{p} \right\rceil \right)^{\lceil K + \frac{d+1}{p} \rceil + 1} \cdot \Omega_0^K \Omega_1 \cdot (1 + R_Q)^{\frac{d}{p}}, & \text{if } p < 1 \end{cases}$$

that

$$\left\| \mathcal{F}^{-1} \left(\varphi_j \cdot \widehat{f} \right) \right\|_{V_j} \leq C_1 \cdot \left\| \mathcal{F}^{-1} \left(\varphi_j \cdot \widehat{f} \right) \right\|_{L_v^p} \quad \forall j \in I \quad \forall f \in Z'(\mathcal{O}),$$

since we have $\text{supp} \left(\varphi_j \cdot \widehat{f} \right) \subset \overline{Q_j} \subset T_j \left[\overline{B_{R_Q}}(0) \right] + b_j \subset T_j \left[-R_Q, R_Q \right]^d + b_j$. This easily shows that the map

$$\text{Ana}_\varphi : \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q) \rightarrow \ell_w^q \left([V_j]_{j \in I} \right), f \mapsto \left(\mathcal{F}^{-1} \left[\varphi_j \cdot \widehat{f} \right] \right)_{j \in I}$$

is well-defined and bounded, with $\|\text{Ana}_\varphi\| \leq C_1$.

Furthermore, Lemma 3.8 shows that the map

$$m_\theta : \ell_w^q \left([V_j]_{j \in I} \right) \rightarrow \ell_w^q \left([V_j]_{j \in I} \right), (f_j)_{j \in I} \mapsto [(\mathcal{F}^{-1} \theta_j) * f_j]_{j \in I} \stackrel{\text{Lem. 3.3}}{=} \left[\mathcal{F}^{-1} \left(\theta_j \cdot \widehat{f_j} \right) \right]_{j \in I}$$

is well-defined and bounded, with $\|m_\theta\| \leq C_2 < \infty$ for

$$C_2 := \begin{cases} \Omega_0^{4K} \Omega_1^4 \Omega_2^{(p,K)} \cdot d^{-\frac{d}{2p}} \cdot (972 \cdot d^{5/2})^{K+\frac{d}{p}}, & \text{if } p \in (0, 1), \\ \Omega_0^K \Omega_1 \Omega_2^{(p,K)}, & \text{if } p \in [1, \infty]. \end{cases}$$

Next, it follows from Lemma 5.2 that the map

$$\text{Synth}_{\Gamma_1} : \ell_w^q \left([V_j]_{j \in I} \right) \rightarrow \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q), (g_j)_{j \in I} \mapsto \sum_{j \in I} \gamma_1^{(j)} * g_j = \sum_{j \in I} \mathcal{F}^{-1} \left(\widehat{\gamma_1^{(j)}} \cdot \widehat{g_j} \right)$$

is well-defined and bounded with $\|\text{Synth}_{\Gamma_1}\| \leq C_3 \cdot \|\vec{C}\|^{\max\{1, 1/p\}}$, with

$$C_3 := \begin{cases} 1, & \text{if } p \geq 1 \\ \frac{(2^6/\sqrt{d})^{\frac{d}{p}}}{2^{21} \cdot d^7} \cdot \left(2^{21} \cdot d^5 \cdot \left\lceil K + \frac{d+1}{p} \right\rceil \right)^{\left\lceil K + \frac{d+1}{p} \right\rceil + 1} \cdot (1 + R_Q)^{\frac{d}{p}} \cdot \Omega_0^{5K} \Omega_1^5, & \text{if } p < 1. \end{cases}$$

Finally, we will show below that the map

$$m_{\Gamma_2} : \ell_w^q \left([V_j]_{j \in I} \right) \rightarrow \ell_w^q \left([V_j]_{j \in I} \right), (f_j)_{j \in I} \mapsto \left(\gamma_2^{(j)} * f_j \right)_{j \in I}$$

is also well-defined and bounded. Once this is shown, note that we have

$$\begin{aligned} (\text{Synth}_{\Gamma_1} \circ m_{\Gamma_2} \circ m_\theta \circ \text{Ana}_\varphi) f &= \sum_{j \in I} \left(\gamma_1^{(j)} * \gamma_2^{(j)} * \mathcal{F}^{-1} \theta_j * \mathcal{F}^{-1} \left(\varphi_j \cdot \widehat{f} \right) \right) \\ &\stackrel{(\text{Lemma 3.3})}{=} \sum_{j \in I} \mathcal{F}^{-1} \left(\widehat{\gamma_1^{(j)}} \cdot \widehat{\gamma_2^{(j)}} \cdot \theta_j \cdot \varphi_j \cdot \widehat{f} \right) \\ &\stackrel{(\text{easy consequence of } \gamma_j = \gamma_{j,1} * \gamma_{j,2})}{=} \sum_{j \in I} \mathcal{F}^{-1} \left(\widehat{\gamma^{(j)}} \cdot \theta_j \cdot \varphi_j \cdot \widehat{f} \right) \\ &\stackrel{(\text{since } \widehat{\gamma^{(j)}} \cdot \theta_j \equiv 1 \text{ on } \overline{Q_j} \supset \text{supp } \varphi_j)}{=} \sum_{j \in I} \mathcal{F}^{-1} \left(\varphi_j \cdot \widehat{f} \right) \\ &\stackrel{(\text{since } \widehat{f} \in \mathcal{D}'(\mathcal{O}) \text{ and } (\varphi_j)_{j \in I} \text{ is locally finite part. of unity on } \mathcal{O})}{=} f \end{aligned} \tag{5.14}$$

for all $f \in \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$. Thus, our goal in the remainder of the proof—once we have shown boundedness of m_{Γ_2} —will be to discretize this **reproducing formula**.

But first of all, let us verify boundedness of m_{Γ_2} . To this end, it suffices to show that each map

$$J_j : V_j \rightarrow V_j, f \mapsto \gamma_2^{(j)} * f$$

is bounded, with $\sup_{j \in I} \|J_j\| < \infty$. But for $p \in [1, \infty]$, this simply follows from the weighted Young inequality (equation (1.12)), since in this case, we have $K_0 = K + d + 1$ and thus (cf. equation (5.2))

$$\begin{aligned}
\left\| \gamma_2^{(j)} \right\|_{L_{v_0}^1} &= \|v_0 \cdot |\det T_j| \cdot M_{b_j} [\gamma_{j,2} \circ T_j^T]\|_{L^1} \\
&= \|(v_0 \circ T_j^{-T}) \cdot \gamma_{j,2}\|_{L^1} \\
&\stackrel{(\text{assump. on } v_0 \text{ and eq. (1.11)})}{\leq} \Omega_0^K \Omega_1 \cdot \left\| (1 + |\bullet|)^K \cdot \gamma_{j,2} \right\|_{L^1} \\
&\leq \Omega_0^K \Omega_1 \cdot \|\gamma_{j,2}\|_{K_0} \cdot \left\| (1 + |\bullet|)^{K-K_0} \right\|_{L^1} \\
&\stackrel{(\text{eq. (1.9)})}{\leq} \Omega_0^K \Omega_1 \Omega_4^{(p,K,1)} \cdot s_d < \infty.
\end{aligned} \tag{5.15}$$

Here, we defined $\Omega_4^{(p,K,1)} := \sup_{j \in I} \|\gamma_{j,2}\|_{K_0}$ in the last step. Note that $\Omega_4^{(p,K,1)} \leq \Omega_4^{(p,K)}$, cf. Assumption 5.1, equation (5.1).

Likewise, for $p \in (0, 1)$, we can simply use Corollary 2.22 to derive for $C_4 := \Omega_0^{3K} \Omega_1^3 d^{-\frac{d}{2p}} \cdot (972 \cdot d^{5/2})^{K+\frac{d}{p}}$ that

$$\begin{aligned}
\left\| \gamma_2^{(j)} * f \right\|_{V_j} &= \left\| \gamma_2^{(j)} * f \right\|_{W_{T_j^{-T}[-1,1]^d}(L_{v_0}^p)} \\
&\stackrel{(\text{Cor. 2.22})}{\leq} C_4 \cdot |\det T_j|^{\frac{1}{p}-1} \cdot \left\| \gamma_2^{(j)} \right\|_{W_{T_j^{-T}[-1,1]^d}(L_{v_0}^p)} \cdot \|f\|_{W_{T_j^{-T}[-1,1]^d}(L_{v_0}^p)} \\
&\stackrel{(\text{eq. (5.2)})}{=} C_4 \cdot |\det T_j|^{\frac{1}{p}} \cdot \|M_{b_j} [\gamma_{j,2} \circ T_j^T]\|_{W_{T_j^{-T}[-1,1]^d}(L_{v_0}^p)} \cdot \|f\|_{V_j} \\
&= C_4 \cdot |\det T_j|^{\frac{1}{p}} \cdot \|v_0 \cdot M_{T_j^{-T}[-1,1]^d} [\gamma_{j,2} \circ T_j^T]\|_{L^p} \cdot \|f\|_{V_j} \\
&\stackrel{(\text{Lemma 2.4})}{=} C_4 \cdot |\det T_j|^{\frac{1}{p}} \cdot \|v_0 \cdot (M_{[-1,1]^d} \gamma_{j,2}) \circ T_j^T\|_{L^p} \cdot \|f\|_{V_j} \\
&= C_4 \cdot \|(v_0 \circ T_j^{-T}) \cdot M_{[-1,1]^d} \gamma_{j,2}\|_{L^p} \cdot \|f\|_{V_j} \\
&\stackrel{(\text{assump. on } v_0 \text{ and eq. (1.11)})}{\leq} \Omega_0^K \Omega_1 \cdot C_4 \cdot \left\| (1 + |\bullet|)^K \cdot M_{[-1,1]^d} \gamma_{j,2} \right\|_{L^p} \cdot \|f\|_{V_j} \\
&\stackrel{(\text{Lemma 2.3})}{\leq} \Omega_0^K \Omega_1 \cdot C_4 \cdot (1 + 2\sqrt{d})^{K_0} \cdot \|\gamma_{j,2}\|_{K_0} \cdot \left\| (1 + |\bullet|)^{K-K_0} \right\|_{L^p} \cdot \|f\|_{V_j} \\
&\stackrel{(\text{eq. (1.9) and } K-K_0=-(\frac{d}{p}+1))}{\leq} \Omega_0^K \Omega_1 \Omega_4^{(p,K,1)} \cdot C_4 \cdot (1 + 2\sqrt{d})^{K_0} \cdot \left(\frac{s_d}{p} \right)^{1/p} \cdot \|f\|_{V_j}.
\end{aligned}$$

Here, we used the same definition of $\Omega_4^{(p,K,1)}$ as above. We have thus established boundedness of m_{Γ_2} in all cases.

In order to discretize the reproducing formula from equation (5.14), we define for $\delta \in (0, 1]$ the map

$$\begin{aligned}
D^{(\delta)} : \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q) &\rightarrow \ell_{\left(|\det T_j|^{\frac{1}{2}-\frac{1}{p}} w_j\right)_{j \in I}}^q \left([C_j^{(\delta)}]_{j \in I} \right), \\
f &\mapsto \left[\left(\delta^d \cdot |\det T_j|^{-1/2} \cdot [\mathcal{F}^{-1}(\theta_j \varphi_j \cdot \widehat{f})](\delta \cdot T_j^{-T} k) \right)_{k \in \mathbb{Z}^d} \right]_{j \in I}.
\end{aligned}$$

This map is indeed well-defined and bounded, since Lemma 5.4 yields for

$$C_5 := 2^{\max\{1, \frac{1}{p}\}} \cdot \Omega_0^{3K} \Omega_1^3 \cdot (1 + \sqrt{d})^K \cdot \left(23040 \cdot d^{3/2} \cdot \left(K + 1 + \frac{d+1}{\min\{1, p\}} \right) \right)^{K+2+\frac{d+1}{\min\{1, p\}}} \cdot (1 + R_{\mathcal{Q}})^{1+\frac{d}{\min\{1, p\}}}$$

that

$$\begin{aligned}
\|D^{(\delta)} f\|_{\ell_w^q \left(|\det T_j|^{\frac{1}{2} - \frac{1}{p}} w_j \right)_{j \in I}^{(C_j^{(\delta)})}} &= \delta^d \cdot \left\| \left(|\det T_j|^{-1/p} \cdot \left\| \left(\mathcal{F}^{-1}(\theta_j \varphi_j \cdot \widehat{f}) \right) (\delta \cdot T_j^{-T} k) \right\|_{C_j^{(\delta)}} \right)_{k \in \mathbb{Z}^d} \right\|_{\ell_w^q(I)} \\
&\leq C_5 \cdot \delta^{d(1-\frac{1}{p})} \cdot \left\| \left(\left\| \mathcal{F}^{-1}(\theta_j \varphi_j \cdot \widehat{f}) \right\|_{L_v^p} \right)_{j \in I} \right\|_{\ell_w^q(I)} \\
&\leq C_5 \cdot \delta^{d(1-\frac{1}{p})} \cdot \left\| \left(\left\| \mathcal{F}^{-1}(\theta_j \varphi_j \cdot \widehat{f}) \right\|_{V_j} \right)_{j \in I} \right\|_{\ell_w^q(I)} \\
&= C_5 \cdot \delta^{d(1-\frac{1}{p})} \cdot \| (m_\theta \circ \text{Ana}_\varphi) f \|_{\ell_w^q([V_j]_{j \in I})} \\
&\leq C_5 \cdot \|m_\theta\| \cdot \|\text{Ana}_\varphi\| \cdot \delta^{d(1-\frac{1}{p})} \cdot \|f\|_{\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)} < \infty.
\end{aligned}$$

Now, our goal is to show for

$$E^{(\delta)} := (m_{\Gamma_2} \circ m_\theta \circ \text{Ana}_\varphi) - \left(\left[\bigotimes_{j \in I} S_{\Gamma_2}^{(\delta, j)} \right] \circ D^{(\delta)} \right) : \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q) \rightarrow \ell_w^q([V_j]_{j \in I})$$

that we have $\|\text{Synth}_{\Gamma_1}\| \cdot \|E^{(\delta)}\| \leq \frac{1}{2}$ for all $0 < \delta \leq \min\{1, \delta_0\}$.

To this end, let $f \in \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$ be arbitrary and for brevity, let

$$f_j := \mathcal{F}^{-1}(\theta_j \varphi_j \widehat{f}) = [(m_\theta \circ \text{Ana}_\varphi) f]_j \in V_j,$$

as well as $f_j^{(2)} := M_{-b_j} f_j$ and $\gamma_2^{(j,2)} := M_{-b_j} \gamma_2^{(j)} = |\det T_j| \cdot \gamma_{j,2} \circ T_j^T$. Note that since $f_j \in V_j$ is bandlimited with $\text{supp } \widehat{f_j} \subset \overline{Q_j} \subset T_j[-R_Q, R_Q]^d + b_j$, Theorem 2.17 and equation (2.13) yield $f_j \in W_{T_j^{-T}[-1,1]^d}(L_v^p) \hookrightarrow L_v^\infty(\mathbb{R}^d)$. Since $\gamma_2^{(j)} \in L_{v_0}^1(\mathbb{R}^d)$, this implies that the integral defining $(\gamma_2^{(j)} * f_j)(x)$ exists for every $x \in \mathbb{R}^d$, cf. equation (2.16). Hence, using our newly introduced notation, we have

$$\begin{aligned}
&\left| [E^{(\delta)} f]_j(x) \right| \\
&= \left| \left[\gamma_2^{(j)} * \mathcal{F}^{-1}(\theta_j \varphi_j \cdot \widehat{f}) \right] (x) - |\det T_j|^{-\frac{1}{2}} \sum_{k \in \mathbb{Z}^d} \delta^d |\det T_j|^{-\frac{1}{2}} \cdot \left[\mathcal{F}^{-1}(\theta_j \varphi_j \cdot \widehat{f}) \right] (\delta \cdot T_j^{-T} k) \cdot \left(L_{\delta \cdot T_j^{-T} k} \gamma_2^{(j)} \right) (x) \right| \\
&= \left| \sum_{k \in \mathbb{Z}^d} \left[\int_{\delta T_j^{-T}(k+[0,1]^d)} \gamma_2^{(j)}(x-y) \cdot f_j(y) \, dy - \delta^d |\det T_j^{-T}| \cdot f_j(\delta \cdot T_j^{-T} k) \cdot \gamma_2^{(j)}(x - \delta \cdot T_j^{-T} k) \right] \right| \\
&\leq \sum_{k \in \mathbb{Z}^d} \int_{\delta T_j^{-T}(k+[0,1]^d)} \left| \gamma_2^{(j)}(x-y) \cdot f_j(y) - f_j(\delta \cdot T_j^{-T} k) \cdot \gamma_2^{(j)}(x - \delta \cdot T_j^{-T} k) \right| \, dy \\
&\stackrel{(*)}{=} \sum_{k \in \mathbb{Z}^d} \int_{\delta T_j^{-T}(k+[0,1]^d)} \left| \gamma_2^{(j,2)}(x-y) \cdot f_j^{(2)}(y) - \gamma_2^{(j,2)}(x - \delta \cdot T_j^{-T} k) f_j^{(2)}(\delta \cdot T_j^{-T} k) \right| \, dy \\
&\leq \sum_{k \in \mathbb{Z}^d} \int_{\delta T_j^{-T}(k+[0,1]^d)} \left| \gamma_2^{(j,2)}(x-y) \left[f_j^{(2)}(y) - f_j^{(2)}(\delta T_j^{-T} k) \right] \right| + \left| f_j^{(2)}(\delta T_j^{-T} k) \left[\gamma_2^{(j,2)}(x-y) - \gamma_2^{(j,2)}(x - \delta T_j^{-T} k) \right] \right| \, dy.
\end{aligned}$$

In this calculation, we used at (*) the easily verifiable identity $(M_b f)(x-y) \cdot (M_b g)(y) = e^{2\pi i \langle b, x \rangle} \cdot f(x-y) g(y)$.

Next, note for arbitrary $y \in \delta T_j^{-T}(k+[0,1]^d)$ that $y = \delta T_j^{-T} k + \delta T_j^{-T} u$ for some $u \in [-1,1]^d$. This implies $\delta T_j^{-T} k = y - \delta T_j^{-T} u \in y + \delta T_j^{-T}[-1,1]^d$ and hence

$$\left| f_j^{(2)}(y) - f_j^{(2)}(\delta \cdot T_j^{-T} k) \right| \leq \left(\text{osc}_{\delta T_j^{-T}[-1,1]^d} f_j^{(2)} \right)(y).$$

Likewise, we have $x - \delta T_j^{-T} k = x - (y - \delta T_j^{-T} u) \in x - y + \delta T_j^{-T}[-1,1]^d$, which yields

$$\left| \gamma_2^{(j,2)}(x-y) - \gamma_2^{(j,2)}(x - \delta \cdot T_j^{-T} k) \right| \leq \left(\text{osc}_{\delta T_j^{-T}[-1,1]^d} \gamma_2^{(j,2)} \right)(x-y).$$

Finally, we also have

$$\begin{aligned} \left| f_j^{(2)}(\delta \cdot T_j^{-T} k) \right| &\leq \left| f_j^{(2)}(\delta \cdot T_j^{-T} k) - f_j^{(2)}(y) \right| + \left| f_j^{(2)}(y) \right| \\ &\leq \left| f_j^{(2)}(y) \right| + \left(\text{osc}_{\delta T_j^{-T}[-1,1]^d} f_j^{(2)} \right)(y) \\ &=: e_j(y), \end{aligned}$$

so that we see

$$\begin{aligned} &\left| \left[E^{(\delta)} f \right]_j(x) \right| \\ &\leq \sum_{k \in \mathbb{Z}^d} \int_{\delta T_j^{-T}(k+[0,1]^d)} \left| \gamma_2^{(j,2)}(x-y) \right| \cdot \left[\text{osc}_{\delta T_j^{-T}[-1,1]^d} f_j^{(2)} \right](y) + e_j(y) \cdot \left[\text{osc}_{\delta T_j^{-T}[-1,1]^d} \gamma_2^{(j,2)} \right](x-y) dy \\ &= \left(\left| \gamma_2^{(j,2)} \right| * \left[\text{osc}_{\delta T_j^{-T}[-1,1]^d} f_j^{(2)} \right] \right)(x) + \left(e_j * \left[\text{osc}_{\delta T_j^{-T}[-1,1]^d} \gamma_2^{(j,2)} \right] \right)(x). \end{aligned} \quad (5.16)$$

We now distinguish two cases: For $p \in [1, \infty]$, first note $|\gamma_2^{(j,2)}| = |\gamma_2^{(j)}|$ and hence, thanks to equation (5.15), $\|\gamma_2^{(j,2)}\|_{L_{v_0}^1} = \|\gamma_2^{(j)}\|_{L_{v_0}^1} \leq \Omega_0^K \Omega_1 \Omega_4^{(p,K,1)} \cdot s_d =: C_6$. Hence, we get using the triangle inequality, the weighted Young inequality (equation (1.12)), the definition of e_j and since $K_0 = K + d + 1$ that

$$\begin{aligned} &\left\| \left[E^{(\delta)} f \right]_j \right\|_{V_j} = \left\| \left[E^{(\delta)} f \right]_j \right\|_{L_v^p} \\ &\stackrel{(\text{def. of } e_j)}{\leq} \left\| \gamma_2^{(j,2)} \right\|_{L_{v_0}^1} \cdot \left\| \text{osc}_{\delta T_j^{-T}[-1,1]^d} f_j^{(2)} \right\|_{L_v^p} + \left\| \text{osc}_{\delta T_j^{-T}[-1,1]^d} \gamma_2^{(j,2)} \right\|_{L_{v_0}^1} \cdot \left(\left\| f_j^{(2)} \right\|_{L_v^p} + \left\| \text{osc}_{\delta T_j^{-T}[-1,1]^d} f_j^{(2)} \right\|_{L_v^p} \right) \\ &\leq C_6 \left\| \text{osc}_{\delta T_j^{-T}[-1,1]^d} [M_{-b_j} f_j] \right\|_{L_v^p} + \left\| \text{osc}_{\delta T_j^{-T}[-1,1]^d} [M_{-b_j} \gamma_2^{(j)}] \right\|_{L_{v_0}^1} \left(\left\| f_j \right\|_{L_v^p} + \left\| \text{osc}_{\delta T_j^{-T}[-1,1]^d} [M_{-b_j} f_j] \right\|_{L_v^p} \right) \\ &\stackrel{(\text{Thm. 2.18})}{\leq} C_6 C_7 \cdot \delta \cdot \left\| f_j \right\|_{L_v^p} + \left\| \text{osc}_{\delta T_j^{-T}[-1,1]^d} [M_{-b_j} \gamma_2^{(j)}] \right\|_{L_{v_0}^1} \cdot \left(\left\| f_j \right\|_{L_v^p} + C_7 \cdot \delta \cdot \left\| f_j \right\|_{L_v^p} \right) \\ &\stackrel{(\text{eq. (5.2)})}{=} C_6 C_7 \cdot \delta \cdot \left\| f_j \right\|_{L_v^p} + |\det T_j| \cdot \left\| \text{osc}_{\delta T_j^{-T}[-1,1]^d} [\gamma_{j,2} \circ T_j^T] \right\|_{L_{v_0}^1} \cdot \left(\left\| f_j \right\|_{L_v^p} + C_7 \cdot \delta \cdot \left\| f_j \right\|_{L_v^p} \right) \\ &\stackrel{(\text{Lem. 2.11})}{=} C_6 C_7 \cdot \delta \cdot \left\| f_j \right\|_{L_v^p} + |\det T_j| \cdot \left\| v_0 \cdot \left(\left[\text{osc}_{\delta[-1,1]^d} \gamma_{j,2} \right] \circ T_j^T \right) \right\|_{L^1} \cdot \left(\left\| f_j \right\|_{L_v^p} + C_7 \cdot \delta \cdot \left\| f_j \right\|_{L_v^p} \right) \\ &\stackrel{(\dagger)}{\leq} C_6 C_7 \cdot \delta \cdot \left\| f_j \right\|_{L_v^p} + \Omega_0^K \Omega_1 \cdot \left\| (1 + |\bullet|)^K \cdot \text{osc}_{\delta[-1,1]^d} \gamma_{j,2} \right\|_{L^1} \cdot \left(\left\| f_j \right\|_{L_v^p} + C_7 \cdot \delta \cdot \left\| f_j \right\|_{L_v^p} \right) \\ &\stackrel{(\text{Lem. 2.14})}{\leq} C_6 C_7 \cdot \delta \cdot \left\| f_j \right\|_{L_v^p} + \Omega_0^K \Omega_1 \cdot \left(3\sqrt{d} \right)^{K_0+1} \cdot \delta \cdot \left\| \nabla \gamma_{j,2} \right\|_{K_0} \cdot \left\| (1 + |\bullet|)^{K-K_0} \right\|_{L^1} \cdot \left(\left\| f_j \right\|_{L_v^p} + C_7 \cdot \delta \cdot \left\| f_j \right\|_{L_v^p} \right) \\ &\stackrel{(\text{eq. (1.9)})}{\leq} C_6 C_7 \cdot \delta \cdot \left\| f_j \right\|_{L_v^p} + \Omega_0^K \Omega_1 \cdot s_d \left(3\sqrt{d} \right)^{K_0+1} \cdot \delta \cdot \left\| \nabla \gamma_{j,2} \right\|_{K_0} \cdot \left(\left\| f_j \right\|_{L_v^p} + C_7 \cdot \delta \cdot \left\| f_j \right\|_{L_v^p} \right) \\ &\stackrel{(\text{since } \delta \leq 1)}{\leq} \delta \cdot \left\| f_j \right\|_{L_v^p} \cdot \left(C_6 C_7 + \Omega_0^K \Omega_1 \cdot s_d \left(3\sqrt{d} \right)^{K_0+1} \cdot \Omega_4^{(p,K,2)} \cdot (1 + C_7) \right) \\ &=: C_8 \cdot \delta \cdot \left\| f_j \right\|_{L_v^p} = C_8 \cdot \delta \cdot \left\| f_j \right\|_{V_j}. \end{aligned}$$

Here, we defined $\Omega_4^{(p,K,2)} := \sup_{j \in I} \|\nabla \gamma_{j,2}\|_{K_0}$, which is finite thanks to equation (5.1). The step marked with (\dagger) used a simple change of variables and our assumption $v_0(x) \leq \Omega_1 \cdot (1 + |x|)^K$ in combination with estimate (1.11). Furthermore, our application of Theorem 2.18 is justified, since we have $f_j = \mathcal{F}^{-1}(\theta_j \varphi_j \hat{f})$, which implies $\text{supp } \hat{f}_j \subset \text{supp } \varphi_j \subset \overline{Q_j} \subset T_j[-R_Q, R_Q]^d + b_j$, so that Theorem 2.18 yields

$$\left\| \text{osc}_{\delta T_j^{-T}[-1,1]^d} [M_{-b_j} f_j] \right\|_{V_j} \leq C_7 \cdot \delta \cdot \left\| f_j \right\|_{L_v^p} \quad (5.17)$$

for

$$C_7 := \Omega_0^{2K} \Omega_1^2 \cdot \left(23040 \cdot d^{\frac{3}{2}} \cdot \left(K + 1 + \frac{d+1}{\min\{1, p\}} \right) \right)^{K+2+\frac{d+1}{\min\{1, p\}}} \cdot (1 + R_Q)^{1+\frac{d}{\min\{1, p\}}} = \left[2^{\max\{1, \frac{1}{p}\}} \Omega_0^K \Omega_1 (1 + \sqrt{d})^K \right]^{-1} \cdot C_5.$$

In case of $p \in (0, 1)$, we let $C_9 := 2^{\frac{1}{p}-1}$, so that C_9 is a triangle constant for $L^p(\mathbb{R}^d)$. Furthermore, we set $V_j^h := W_{T_j^{-T}[-1,1]^d}(L_{v_0}^p)$ for brevity. Then, we use Corollary 2.22 to get for $C_{10} := C_9 \cdot \Omega_0^{3K} \Omega_1^3 \cdot d^{-\frac{d}{2p}} \cdot (972 \cdot d^{5/2})^{K+\frac{d}{p}}$

that

$$\begin{aligned}
& \left\| \left[E^{(\delta)} f \right]_j \right\|_{V_j} \\
& \stackrel{(\text{eq. (5.16)})}{\leq} C_9 \cdot \left[\left\| \left| \gamma_2^{(j,2)} \right| * \text{osc}_{\delta T_j^{-T}[-1,1]^d} f_j^{(2)} \right\|_{V_j} + \left\| e_j * \left[\text{osc}_{\delta T_j^{-T}[-1,1]^d} \gamma_2^{(j,2)} \right] \right\|_{V_j} \right] \\
& \stackrel{(\text{Cor. 2.22})}{\leq} C_{10} \cdot |\det T_j|^{\frac{1}{p}-1} \left(\left\| \gamma_2^{(j,2)} \right\|_{V_j^\sharp} \cdot \left\| \text{osc}_{\delta T_j^{-T}[-1,1]^d} f_j^{(2)} \right\|_{V_j} + \|e_j\|_{V_j} \cdot \left\| \text{osc}_{\delta T_j^{-T}[-1,1]^d} \gamma_2^{(j,2)} \right\|_{V_j^\sharp} \right) \\
& \stackrel{(\text{eq. (5.2), def. of } e_j)}{\leq} C_{10} \cdot |\det T_j|^{\frac{1}{p}} \cdot \left(\left\| \gamma_{j,2} \circ T_j^T \right\|_{V_j^\sharp} \cdot \left\| \text{osc}_{\delta T_j^{-T}[-1,1]^d} [M_{-b_j} f_j] \right\|_{V_j} \right. \\
& \quad \left. + C_9 \left[\|f_j\|_{V_j} + \left\| \text{osc}_{\delta T_j^{-T}[-1,1]^d} [M_{-b_j} f_j] \right\|_{V_j} \right] \cdot \left\| \text{osc}_{\delta T_j^{-T}[-1,1]^d} [\gamma_{j,2} \circ T_j^T] \right\|_{V_j^\sharp} \right) \\
& \stackrel{(\text{Lem. 2.11, eq. (5.17)})}{\leq} C_{10} \cdot |\det T_j|^{\frac{1}{p}} \cdot \left(\left\| v_0 \cdot M_{T_j^{-T}[-1,1]^d} [\gamma_{j,2} \circ T_j^T] \right\|_{L^p} \cdot C_7 \cdot \delta \cdot \|f_j\|_{V_j} \right. \\
& \quad \left. + C_9 \left[\|f_j\|_{V_j} + C_7 \cdot \delta \cdot \|f_j\|_{V_j} \right] \cdot \left\| v_0 \cdot M_{T_j^{-T}[-1,1]^d} \left[\left(\text{osc}_{\delta[-1,1]^d} \gamma_{j,2} \right) \circ T_j^T \right] \right\|_{L^p} \right) \\
& \stackrel{(\text{Lem. 2.4})}{=} C_{10} \cdot \left(\left\| (v_0 \circ T_j^{-T}) \cdot M_{[-1,1]^d} \gamma_{j,2} \right\|_{L^p} \cdot C_7 \cdot \delta \cdot \|f_j\|_{V_j} \right. \\
& \quad \left. + C_9 \left[\|f_j\|_{V_j} + C_7 \cdot \delta \cdot \|f_j\|_{V_j} \right] \cdot \left\| (v_0 \circ T_j^{-T}) \cdot M_{[-1,1]^d} \left[\text{osc}_{\delta[-1,1]^d} \gamma_{j,2} \right] \right\|_{L^p} \right) \\
& \stackrel{(\text{since } \delta \leq 1)}{\leq} C_{10} \Omega_0^K \Omega_1 \cdot \left(\left\| (1 + |\bullet|)^K \cdot M_{[-1,1]^d} \gamma_{j,2} \right\|_{L^p} \cdot C_7 \cdot \delta \cdot \|f_j\|_{V_j} \right. \\
& \quad \left. + C_9 \|f_j\|_{V_j} (1 + C_7) \cdot \left\| (1 + |\bullet|)^K \cdot M_{[-1,1]^d} \left[\text{osc}_{\delta[-1,1]^d} \gamma_{j,2} \right] \right\|_{L^p} \right).
\end{aligned}$$

Here, the last step used as usual our assumption $v_0(x) \leq \Omega_1 \cdot (1 + |x|)^K$, in combination with equation (1.11). We now combine Lemmas 2.14 and 2.3 to obtain

$$\begin{aligned}
& \left\| (1 + |\bullet|)^K \cdot M_{[-1,1]^d} \left[\text{osc}_{\delta[-1,1]^d} \gamma_{j,2} \right] \right\|_{L^p} \leq \left(3\sqrt{d} \right)^{K_0+1} \cdot \delta \cdot \|\nabla \gamma_{j,2}\|_{K_0} \cdot \left\| (1 + |\bullet|)^K \cdot M_{[-1,1]^d} (1 + |\bullet|)^{-K_0} \right\|_{L^p} \\
& \leq \left(1 + 2\sqrt{d} \right)^{K_0} \left(3\sqrt{d} \right)^{K_0+1} \cdot \delta \cdot \|\nabla \gamma_{j,2}\|_{K_0} \cdot \left\| (1 + |\bullet|)^{K-K_0} \right\|_{L^p} \\
& \stackrel{(\text{eq. (1.9)})}{\leq} \left(3\sqrt{d} \right)^{2K_0+1} \Omega_4^{(p,K,2)} \cdot \delta \cdot \left(\frac{s_d}{p} \right)^{1/p}.
\end{aligned}$$

Likewise, Lemma 2.3 also yields

$$\begin{aligned}
& \left\| (1 + |\bullet|)^K \cdot M_{[-1,1]^d} \gamma_{j,2} \right\|_{L^p} \leq \|\gamma_{j,2}\|_{K_0} \cdot \left\| (1 + |\bullet|)^K \cdot M_{[-1,1]^d} (1 + |\bullet|)^{-K_0} \right\|_{L^p} \\
& \leq \|\gamma_{j,2}\|_{K_0} \cdot \left(1 + 2\sqrt{d} \right)^{K_0} \cdot \left\| (1 + |\bullet|)^{K-K_0} \right\|_{L^p} \\
& \leq \left(3\sqrt{d} \right)^{K_0} \Omega_4^{(p,K,1)} \cdot \left(\frac{s_d}{p} \right)^{1/p}.
\end{aligned}$$

Combining these estimates with our estimate for $\left\| [E^{(\delta)} f]_j \right\|_{V_j}$, we arrive at

$$\begin{aligned}
& \left\| [E^{(\delta)} f]_j \right\|_{V_j} \leq C_{10} \left(\frac{s_d}{p} \right)^{1/p} \left(3\sqrt{d} \right)^{2K_0+1} \cdot \Omega_0^K \Omega_1 \cdot \left(C_7 \Omega_4^{(p,K,1)} + C_9 (1 + C_7) \Omega_4^{(p,K,2)} \right) \cdot \delta \cdot \|f_j\|_{V_j} \\
& =: C_{11} \cdot \delta \cdot \|f_j\|_{V_j},
\end{aligned}$$

where C_{11} is independent of δ and j .

All in all, if we set $C_{12} := C_8$ for $p \in [1, \infty]$ and $C_{12} := C_{11}$ for $p \in (0, 1)$, we have $\left\| [E^{(\delta)} f]_j \right\|_{V_j} \leq C_{12} \cdot \delta \cdot \|f_j\|_{V_j}$ for all $j \in I$ and $\delta \in (0, 1]$. But this entails

$$\begin{aligned} \left\| E^{(\delta)} f \right\|_{\ell_w^q([V_j]_{j \in I})} &\leq C_{12} \cdot \delta \cdot \left\| (f_j)_{j \in I} \right\|_{\ell_w^q([V_j]_{j \in I})} \\ (\text{since } f_j &= [(m_\theta \circ \text{Ana}_\varphi) f]_j) = C_{12} \cdot \delta \cdot \|(m_\theta \circ \text{Ana}_\varphi) f\|_{\ell_w^q([V_j]_{j \in I})} \\ &\leq C_{12} \cdot \|m_\theta\| \cdot \|\text{Ana}_\varphi\| \cdot \delta \cdot \|f\|_{\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)} \\ &\leq C_1 C_2 C_{12} \cdot \delta \cdot \|f\|_{\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)}. \end{aligned}$$

But since $\|\text{Synth}_{\Gamma_1}\| \leq C_3 \cdot \|\vec{C}\|^{\max\{1, \frac{1}{p}\}}$, this means in view of equation (5.14) that

$$\begin{aligned} \left\| \text{id}_{\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)} - \text{Synth}_{\Gamma_1} \circ \bigotimes_{j \in I} S_{\Gamma_2}^{(\delta, j)} \circ D^{(\delta)} \right\| &= \left\| \text{Synth}_{\Gamma_1} \circ m_{\Gamma_2} \circ m_\theta \circ \text{Ana}_\varphi - \text{Synth}_{\Gamma_1} \circ \bigotimes_{j \in I} S_{\Gamma_2}^{(\delta, j)} \circ D^{(\delta)} \right\| \\ &\leq \|\text{Synth}_{\Gamma_1}\| \cdot \|E^{(\delta)}\| \\ &\leq C_1 C_2 C_3 C_{12} \cdot \|\vec{C}\|^{\max\{1, \frac{1}{p}\}} \cdot \delta. \end{aligned}$$

Now, we estimate the constant $C_1 C_2 C_3 C_{12}$ to see that $\delta \leq \delta_0$ implies $C_1 C_2 C_3 C_{12} \cdot \|\vec{C}\|^{\max\{1, \frac{1}{p}\}} \cdot \delta \leq \frac{1}{2}$. First, in case of $p \in [1, \infty]$, we have because of $C_7 \geq 1$ and $\max\{1, \frac{1}{p}\} = 1$, as well as $K_0 = K + d + 1$ that

$$\begin{aligned} C_1 C_2 C_3 C_{12} &= \Omega_0^K \Omega_1 \Omega_2^{(p, K)} \cdot C_8 \\ &= \left(C_6 C_7 + \Omega_0^K \Omega_1 \cdot s_d \left(3\sqrt{d} \right)^{K_0+1} \cdot \Omega_4^{(p, K, 2)} \cdot (1 + C_7) \right) \cdot \Omega_0^K \Omega_1 \Omega_2^{(p, K)} \\ (\text{since } \Omega_0, \Omega_1 &\geq 1) \leq C_7 s_d \left(\Omega_4^{(p, K, 1)} + 2 \cdot \left(3\sqrt{d} \right)^{K_0+1} \Omega_4^{(p, K, 2)} \right) \cdot \Omega_0^{2K} \Omega_1^2 \Omega_2^{(p, K)} \\ &\leq (2^{17} \cdot d^2 \cdot (K+2+d))^{K+d+3} \cdot 2s_d \cdot \left(3 \cdot d^{\frac{1}{2}} \right)^{-1} \cdot (1 + R_{\mathcal{Q}})^{d+1} \cdot \left(\Omega_4^{(p, K, 1)} + \Omega_4^{(p, K, 2)} \right) \cdot \Omega_0^{4K} \Omega_1^4 \Omega_2^{(p, K)} \\ &\leq \frac{s_d}{\sqrt{d}} \cdot (2^{17} \cdot d^2 \cdot (K+2+d))^{K+d+3} \cdot (1 + R_{\mathcal{Q}})^{d+1} \cdot \Omega_0^{4K} \Omega_1^4 \Omega_2^{(p, K)} \Omega_4^{(p, K)}. \end{aligned}$$

Next, for $p \in (0, 1)$, we get because of $\max\{1, \frac{1}{p}\} = \frac{1}{p}$ and $s_d \leq 2^{2d}$ that

$$\begin{aligned} C_1 C_3 &= \frac{\left(2^{10}/d^{\frac{1}{2}} \right)^{\frac{d}{p}}}{2^{21} \cdot d^7} \cdot 2^4 s_d^{\frac{1}{p}} \left(192 \cdot d^{\frac{3}{2}} \cdot \left\lceil K + \frac{d+1}{p} \right\rceil \right)^{\left\lceil K + \frac{d+1}{p} \right\rceil + 1} \cdot \left(2^{21} \cdot d^5 \cdot \left\lceil K + \frac{d+1}{p} \right\rceil \right)^{\left\lceil K + \frac{d+1}{p} \right\rceil + 1} \cdot (1 + R_{\mathcal{Q}})^{\frac{2d}{p}} \cdot \Omega_0^{6K} \Omega_1^6 \\ &\leq 2^4 \cdot \frac{\left(2^{12}/\sqrt{d} \right)^{\frac{d}{p}}}{2^{21} \cdot d^7} \cdot \left(2^{29} \cdot d^{\frac{13}{2}} \cdot \left\lceil K + \frac{d+1}{p} \right\rceil^2 \right)^{\left\lceil K + \frac{d+1}{p} \right\rceil + 1} \cdot (1 + R_{\mathcal{Q}})^{\frac{2d}{p}} \cdot \Omega_0^{6K} \Omega_1^6 \end{aligned}$$

and thus, since $\left\lceil K + \frac{d+1}{p} \right\rceil + 1 \geq K + \frac{d}{p} + 2$,

$$\begin{aligned} C_1 C_2 C_3 &\leq d^{-\frac{d}{2p}} \cdot \left(972 \cdot d^{\frac{5}{2}} \right)^{-2} \cdot 2^4 \cdot \frac{\left(2^{12}/\sqrt{d} \right)^{\frac{d}{p}}}{2^{21} \cdot d^7} \cdot \left(2^{39} \cdot d^9 \cdot \left\lceil K + \frac{d+1}{p} \right\rceil^2 \right)^{\left\lceil K + \frac{d+1}{p} \right\rceil + 1} \cdot (1 + R_{\mathcal{Q}})^{\frac{2d}{p}} \cdot \Omega_0^{10K} \Omega_1^{10} \Omega_2^{(p, K)} \\ &\leq \frac{\left(2^{12}/d \right)^{\frac{d}{p}}}{2^{36} \cdot d^{12}} \cdot \left(2^{39} \cdot d^9 \cdot \left\lceil K + \frac{d+1}{p} \right\rceil^2 \right)^{\left\lceil K + \frac{d+1}{p} \right\rceil + 1} \cdot (1 + R_{\mathcal{Q}})^{\frac{2d}{p}} \cdot \Omega_0^{10K} \Omega_1^{10} \Omega_2^{(p, K)}. \end{aligned}$$

Now, recall that $C_7 \geq 1$ and $K_0 = K + \frac{d}{p} + 1$, so that

$$\begin{aligned} C_{12} &= C_{11} = C_{10} \left(\frac{s_d}{p} \right)^{1/p} (3\sqrt{d})^{2K_0+1} \cdot \Omega_0^K \Omega_1 \cdot \left(C_7 \Omega_4^{(p,K,1)} + C_9 \Omega_4^{(p,K,2)} (1 + C_7) \right) \\ &\leq C_7 \cdot 2^{\frac{1}{p}-1} \cdot d^{-\frac{d}{2p}} \cdot (972 \cdot d^{5/2})^{K+\frac{d}{p}} \cdot \left(\frac{s_d}{p} \right)^{1/p} (9d)^{K_0+1} \cdot \Omega_0^{4K} \Omega_1^4 \cdot \left(\Omega_4^{(p,K,1)} + 2^{\frac{1}{p}} \Omega_4^{(p,K,2)} \right) \\ &\leq \frac{1}{2} C_7 \cdot 4^{\frac{1}{p}} \cdot d^{-\frac{d}{2p}} \cdot (972 \cdot d^{5/2})^{-2} \cdot (8748 \cdot d^{7/2})^{K_0+1} \cdot \left(\frac{s_d}{p} \right)^{1/p} \cdot \Omega_0^{4K} \Omega_1^4 \Omega_4^{(p,K)} \end{aligned}$$

and hence because of $K_0 + 1 = K + \frac{d}{p} + 2 \leq \left\lceil K + \frac{d+1}{p} \right\rceil + 1$,

$$\begin{aligned} &2C_1 C_2 C_3 C_{12} \\ &\leq C_7 \cdot \frac{(2^{14}/d^{\frac{3}{2}})^{\frac{d}{p}}}{2^{45} \cdot d^{17}} \cdot \left(\frac{s_d}{p} \right)^{\frac{1}{p}} \cdot \left(2^{53} \cdot d^{\frac{25}{2}} \cdot \left\lceil K + \frac{d+1}{p} \right\rceil^2 \right)^{\left\lceil K + \frac{d+1}{p} \right\rceil + 1} \cdot (1 + R_{\mathcal{Q}})^{\frac{2d}{p}} \cdot \Omega_0^{14K} \Omega_1^{14} \Omega_2^{(p,K)} \Omega_4^{(p,K)} \\ &\leq \frac{(2^{14}/d^{\frac{3}{2}})^{\frac{d}{p}}}{2^{45} \cdot d^{17}} \cdot \left(\frac{s_d}{p} \right)^{\frac{1}{p}} \cdot \left(2^{68} \cdot d^{14} \cdot \left\lceil K + 1 + \frac{d+1}{p} \right\rceil^3 \right)^{K+\frac{d+1}{p}+2} \cdot (1 + R_{\mathcal{Q}})^{1+\frac{3d}{p}} \cdot \Omega_0^{16K} \Omega_1^{16} \Omega_2^{(p,K)} \Omega_4^{(p,K)}. \end{aligned}$$

These considerations easily show that $\delta \leq \delta_0$ indeed implies $C_1 C_2 C_3 C_{12} \cdot \|\vec{C}\|^{\max\{1, \frac{1}{p}\}} \cdot \delta \leq \frac{1}{2}$.

All in all, our considerations show for

$$T^{(\delta)} := \text{Synth}_{\Gamma_1} \circ \bigotimes_{j \in I} S_{\Gamma_2}^{(\delta, j)} \circ D^{(\delta)} \stackrel{\text{eq. (5.13)}}{=} S^{(\delta)} \circ D^{(\delta)} : \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q) \rightarrow \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$$

that $\|\text{id}_{\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)} - T^{(\delta)}\| \leq \frac{1}{2}$ for all $0 < \delta \leq \min\{1, \delta_0\}$. Hence, since $\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$ is a Quasi-Banach space by Lemma 5.5, a Neumann series argument (which is also valid for Quasi-Banach spaces, cf. e.g. [76, Lemma 2.4.11]), shows that $T^{(\delta)}$ is invertible for all $0 < \delta \leq \min\{1, \delta_0\}$.

But then, $C^{(\delta)} := D^{(\delta)} \circ (T^{(\delta)})^{-1} : \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q) \rightarrow \ell^q_{\left(|\det T_j|^{\frac{1}{2} - \frac{1}{p} w_j}\right)_{j \in I}} \left([C_j^{(\delta)}]_{j \in I}\right)$ is well-defined and bounded and we have for arbitrary $f \in \mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$ that

$$f = \left[T^{(\delta)} \circ (T^{(\delta)})^{-1} \right] f = \left(\left[S^{(\delta)} \circ D^{(\delta)} \right] \circ (T^{(\delta)})^{-1} \right) f = \left[S^{(\delta)} \circ C^{(\delta)} \right] f,$$

as desired. \square

6. SIMPLIFIED CRITERIA

In this section, we will derive simplified conditions which ensure boundedness of the operators \vec{A}, \vec{B} and \vec{C} , mentioned in Assumptions 3.1, 4.1 and 5.1, respectively.

One such general criterion is given by **Schur's test**, which we state below. Afterwards, we will provide a convenient standard estimate for the main term $\left\| \mathcal{F}^{-1} \left(\varphi_i \cdot \widehat{\gamma^{(j)}} \right) \right\|_{L_{v_0}^p}$ occurring in the entries of \vec{A}, \vec{B} and \vec{C} .

Then we use these results to formulate simplified criteria which allow to apply Theorems 4.7 (leading to Banach frames) and 5.6 (leading to atomic decompositions).

But first of all, we introduce certain additional assumptions regarding the partition of unity $\Phi = (\varphi_i)_{i \in I}$. Recall that in the preceding sections, we only assumed Φ to be a \mathcal{Q} - v_0 -BAPU, but in this section we will make the following stronger assumption:

Assumption 6.1. We assume that $\Phi = (\varphi_i)_{i \in I}$ is a **regular partition of unity** for \mathcal{Q} . This means

- (1) $\varphi_i \in C_c^\infty(\mathcal{O})$ with $\text{supp } \varphi_i \subset Q_i$ for all $i \in I$,
- (2) $\sum_{i \in I} \varphi_i \equiv 1$ on \mathcal{O} ,
- (3) the **normalized family** $\Phi^\natural := (\varphi_i^\natural)_{i \in I}$ —given by $\varphi_i^\natural := \varphi_i \circ S_i$ for $S_i \xi := T_i \xi + b_i$ —satisfies

$$C^{(\alpha)} := \sup_{i \in I} \|\partial^\alpha \varphi_i^\natural\|_{\text{sup}} < \infty \quad \text{for all } \alpha \in \mathbb{N}_0^d. \quad (6.1)$$

Remark. As seen in [77, Lemma 2.5], every regular partition of unity is also a \mathcal{Q} - v_0 -BAPU, as long as $v_0 \lesssim 1$. As we will see in Corollary 6.5, the same also remains true for general v_0 .

Furthermore, it was shown in [78, Theorem 2.8] that every *structured* admissible covering \mathcal{Q} admits a regular partition of unity. Here, the semi-structured covering $\mathcal{Q} = (T_i Q'_i + b_i)_{i \in I}$ is called **structured** if $Q'_i = Q$ for all $i \in I$ and some fixed open set $Q \subset \mathbb{R}^d$ and if additionally, there is an open set $P \subset \mathbb{R}^d$, compactly contained in Q , such that the family $(T_i P + b_i)_{i \in I}$ covers all of \mathcal{O} . \blacklozenge

Now that we have clarified our assumptions for this section, we state a version of Schur's test which is suitable for our setting. We remark that this lemma is in no way new; for example, it already appears in [46, Lemma 4.4].

Lemma 6.2. *Let $I, J \neq \emptyset$ be two nonempty sets and let $A = (A_{i,j})_{(i,j) \in I \times J} \in \mathbb{C}^{I \times J}$. Let $p \in (1, \infty)$ and assume that*

$$C_1 := \sup_{i \in I} \sum_{j \in J} |A_{i,j}| \quad \text{and} \quad C_2 := \sup_{j \in J} \sum_{i \in I} |A_{i,j}|$$

are finite. Then the operator

$$\vec{A} : \ell^p(J) \rightarrow \ell^p(I), (c_j)_{j \in J} \mapsto \left(\sum_{j \in J} A_{i,j} c_j \right)_{i \in I}$$

is well-defined and bounded with $\|\vec{A}\| \leq \max\{C_1, C_2\}$.

In case of $p \in (0, 1]$, it suffices if

$$C_3^{(p)} := \sup_{j \in J} \sum_{i \in I} |A_{i,j}|^p$$

is finite. In this case, $\|\vec{A}\| \leq (C_3^{(p)})^{1/p}$.

Finally, in case of $p = \infty$, it suffices if

$$C_4 := \sup_{i \in I} \sum_{j \in J} |A_{i,j}|$$

is finite. In this case, $\|\vec{A}\| \leq C_4$. \blacktriangleleft

Proof. The statement for $p \in (1, \infty)$ follows from the more general form of Schur's test as given e.g. in [29, Theorem 6.18], by considering I and J as measure spaces by equipping them with the counting measure. Strictly speaking, that lemma assumes the underlying measure spaces to be σ -finite (i.e., I, J have to be countable), but since Tonelli's theorem is applicable to uncountable sets equipped with the counting measure, the proof given in [29] still works even for uncountable I, J .

Now, let us assume $p \in (0, 1]$. In this case, we have

$$\begin{aligned} \left\| \vec{A} (c_j)_{j \in J} \right\|_{\ell^p}^p &= \sum_{i \in I} \left| \left(\vec{A} (c_j)_{j \in J} \right)_i \right|^p = \sum_{i \in I} \left| \sum_{j \in J} A_{i,j} \cdot c_j \right|^p \\ &\quad \left(\text{since } (\sum a_j)^p \leq \sum a_j^p \text{ for } p \in (0, 1] \text{ and } a_j \geq 0 \right) \leq \sum_{i \in I} \sum_{j \in J} |A_{i,j}|^p |c_j|^p \\ &= \sum_{j \in J} \left(|c_j|^p \sum_{i \in I} |A_{i,j}|^p \right) \\ &\leq C_3^{(p)} \cdot \sum_{j \in J} |c_j|^p \\ &= C_3^{(p)} \cdot \left\| (c_j)_{j \in J} \right\|_{\ell^p}^p < \infty, \end{aligned}$$

so that $\vec{A} : \ell^p(J) \rightarrow \ell^p(I)$ is bounded with $\|\vec{A}\|_{\ell^p \rightarrow \ell^p} \leq (C_3^{(p)})^{1/p}$.

Finally, let $p = \infty$. For arbitrary $i \in I$, we have

$$\begin{aligned} \left| \left(\vec{A}(c_j)_{j \in J} \right)_i \right| &= \left| \sum_{j \in J} A_{i,j} \cdot c_j \right| \\ &\leq \sum_{j \in J} (|A_{i,j}| \cdot |c_j|) \\ &\leq \left\| (c_j)_{j \in J} \right\|_{\ell^\infty} \cdot \sum_{j \in J} |A_{i,j}| \\ &\leq C_4 \cdot \left\| (c_j)_{j \in J} \right\|_{\ell^p}. \end{aligned}$$

As a further remark, we observe that the case $p \in (1, \infty)$ can be obtained by complex interpolation (i.e., by the Riesz-Thorin Theorem [29, Theorem 6.27]) from the cases $p = 1$ and $p = \infty$, since $C_1 = C_4$ and $C_2 = C_3^{(1)}$. \square

In Lemma 6.4 below, we will derive a convenient estimate for the main term of the “infinite matrices” A, B, C from Assumptions 3.1, 4.1 and 5.1, namely for the term $\left\| \mathcal{F}^{-1} \left(\varphi_i \widehat{\gamma^{(j)}} \right) \right\|_{L_{v_0}^p}$. To derive this estimate, the following lemma will be useful. It makes precise the notion that *smoothness of f yields decay of \widehat{f}* . The statement itself is probably folklore, so no originality is claimed.

Lemma 6.3. *Let $N \in \mathbb{N}_0$ and $g \in W^{N,1}(\mathbb{R}^d)$. Then*

$$(1 + |x|)^N \cdot |\mathcal{F}^{-1}g(x)| \leq (1 + d)^N \cdot \left(|\mathcal{F}^{-1}g(x)| + \sum_{m=1}^d |[\mathcal{F}^{-1}(\partial_m^N g)](x)| \right) \quad \forall x \in \mathbb{R}^d. \quad (6.2) \quad \blacktriangleleft$$

Remark. Here, $W^{N,1}(\mathbb{R}^d)$ is the **Sobolev space** of all functions $g \in L^1(\mathbb{R}^d)$ for which all weak derivatives $\partial^\alpha g$ with $|\alpha| \leq N$ satisfy $\partial^\alpha g \in L^1(\mathbb{R}^d)$. It is a Banach space when equipped with the norm $\|g\|_{W^{N,1}} := \sum_{|\alpha| \leq N} \|\partial^\alpha g\|_{L^1}$. For $N = 0$, we use the convention $W^{N,1}(\mathbb{R}^d) = L^1(\mathbb{R}^d)$. \blacklozenge

Proof. It is well-known (see e.g. [1, Corollary 3.23]) that $C_c^\infty(\mathbb{R}^d) \subset W^{N,1}(\mathbb{R}^d)$ is dense. Furthermore, since $\mathcal{F}^{-1} : L^1(\mathbb{R}^d) \rightarrow C_0(\mathbb{R}^d)$ is well-defined and bounded, where the space $C_0(\mathbb{R}^d)$ of continuous functions vanishing at infinity is equipped with the norm $\|h\|_{\sup} := \sup_{x \in \mathbb{R}^d} |h(x)|$, it is not hard to see that $\mathcal{F}^{-1}g(x)$ and $[\mathcal{F}^{-1}(\partial_m^N g)](x)$ all depend continuously on $g \in W^{N,1}(\mathbb{R}^d)$, for arbitrary $x \in \mathbb{R}^d$ and $m \in \underline{d}$. Hence, we can without loss of generality assume $g \in C_c^\infty(\mathbb{R}^d)$.

But under this assumption, we have (see e.g. [29, Theorem 8.22]) the standard identity

$$[\mathcal{F}^{-1}(\partial_m^N g)](x) = (-2\pi i x_m)^N \cdot (\mathcal{F}^{-1}g)(x) \quad \forall x \in \mathbb{R}^d.$$

In particular, since

$$\left(\sum_{i=1}^k a_i \right)^N \leq (k \cdot \max \{a_i \mid i \in \underline{k}\})^N \leq k^N \cdot \sum_{i=1}^k a_i^N$$

holds for arbitrary $a_1, \dots, a_k \geq 0$ and because of $|x| \leq \|x\|_1$, we get

$$\begin{aligned} (1 + |x|)^N \cdot |(\mathcal{F}^{-1}g)(x)| &\leq \left(1 + \sum_{m=1}^d |x_m| \right)^N \cdot |(\mathcal{F}^{-1}g)(x)| \\ &\leq (d+1)^N \cdot |(\mathcal{F}^{-1}g)(x)| \cdot \left(1 + \sum_{m=1}^d |x_m^N| \right) \\ &= (d+1)^N \cdot \left(|(\mathcal{F}^{-1}g)(x)| + \sum_{m=1}^d \left| \frac{[\mathcal{F}^{-1}(\partial_m^N g)](x)}{(2\pi)^N} \right| \right) \\ &\leq (1+d)^N \cdot \left(|(\mathcal{F}^{-1}g)(x)| + \sum_{m=1}^d |[\mathcal{F}^{-1}(\partial_m^N g)](x)| \right) \end{aligned}$$

for arbitrary $x \in \mathbb{R}^d$, as desired. \square

Now, we are finally in a position to derive the promised estimate for $\left\| \mathcal{F}^{-1} \left(\widehat{\varphi_i \gamma^{(j)}} \right) \right\|_{L_{v_0}^p}$.

Lemma 6.4. *Suppose that $(\varphi_i)_{i \in I}$ satisfies Assumption 6.1. Let $\gamma \in L^1(\mathbb{R}^d)$ and assume that $\widehat{\gamma} \in C^\infty(\mathbb{R}^d)$. For $j \in I$, define*

$$\gamma^{[j]} := \mathcal{F}^{-1}(\widehat{\gamma} \circ S_j^{-1}) = |\det T_j| \cdot M_{b_j}[\gamma \circ T_j^T].$$

Then we have for arbitrary $\varepsilon > 0$, $i, j \in I$ and $p \in (0, \infty)$ the estimate

$$\begin{aligned} \left\| \mathcal{F}^{-1} \left(\widehat{\gamma^{[j]}} \cdot \varphi_i \right) \right\|_{L_{v_0}^p} &\leq C_0 \cdot (1 + \|T_j^{-1} T_i\|)^{\left\lceil K + \frac{d+\varepsilon}{p} \right\rceil} \cdot |\det T_i|^{-\frac{1}{p}} \cdot \int_{Q_i} \max_{|\alpha| \leq \left\lceil K + \frac{d+\varepsilon}{p} \right\rceil} |(\partial^\alpha \widehat{\gamma})(S_j^{-1} \eta)| \, d\eta \\ &\quad (\xi = S_i^{-1} \eta) \leq C_0 \cdot (1 + \|T_j^{-1} T_i\|)^{\left\lceil K + \frac{d+\varepsilon}{p} \right\rceil} \cdot |\det T_i|^{1-\frac{1}{p}} \cdot \int_{Q'_i} \max_{|\alpha| \leq \left\lceil K + \frac{d+\varepsilon}{p} \right\rceil} |(\partial^\alpha \widehat{\gamma})(S_j^{-1} S_i \xi)| \, d\xi, \end{aligned}$$

with

$$C_0 := \Omega_0^K \Omega_1 \cdot (4 \cdot d)^{1+2\left\lceil K + \frac{d+\varepsilon}{p} \right\rceil} \cdot \left(\frac{s_d}{\varepsilon} \right)^{1/p} \cdot \max_{|\alpha| \leq \left\lceil K + \frac{d+\varepsilon}{p} \right\rceil} C^{(\alpha)},$$

where the constants $C^{(\alpha)}$ are defined in Assumption 6.1, equation (6.1). ◀

Remark. With the notation $\gamma^{[j]}$, the usual notation $\gamma^{(j)}$ for a family $\Gamma = (\gamma_i)_{i \in I}$ takes the form $\gamma^{(j)} = \gamma_j^{[j]}$. ◆

Proof. Set $N := \left\lceil K + \frac{d+\varepsilon}{p} \right\rceil$. Now, recall from [78, Lemma 2.6] the identity

$$(\partial^\alpha [f \circ A])(x) = \sum_{\ell_1, \dots, \ell_k \in \underline{d}} [A_{\ell_1, i_1} \cdots A_{\ell_k, i_k} \cdot (\partial_{\ell_1} \cdots \partial_{\ell_k} f)(Ax)] \quad \forall x \in \mathbb{R}^d$$

for arbitrary $A \in \text{GL}(\mathbb{R}^d)$, $k \in \mathbb{N}$, $f \in C^k(\mathbb{R}^d)$ and $\alpha = \sum_{m=1}^k e_{i_m} \in \mathbb{N}_0^d$, where (e_1, \dots, e_d) is the standard basis of \mathbb{R}^d . In particular, this implies for arbitrary $k \in \mathbb{N}$ that

$$|(\partial^\alpha [f \circ A])(x)| \leq d^k \cdot \|A\|^k \cdot \max_{|\beta|=k} |(\partial^\beta f)(Ax)| \quad \forall x \in \mathbb{R}^d \, \forall \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| = k \text{ and } f \in C^k(\mathbb{R}^d)$$

and this estimate obviously also holds for $k = 0$.

Thus, using the notation $h^\heartsuit(\xi) := \max_{|\alpha| \leq N} |(\partial^\alpha h)(\xi)|$ for $\xi \in \mathbb{R}^d$ and $h \in C^\infty(\mathbb{R}^d)$, we get for arbitrary $i \in I$, $m \in \underline{d}$ and $\ell \in \{0\} \cup \underline{N} = \{0, \dots, \left\lceil K + \frac{d+\varepsilon}{p} \right\rceil\}$ as well as $T \in \text{GL}(\mathbb{R}^d)$ that

$$\begin{aligned} |[\partial_m^\ell (h \circ S_i^{-1} \circ T)](\xi)| &= |[\partial_m^\ell (\eta \mapsto h(T_i^{-1} T \eta - T_i^{-1} b_i))](\xi)| \\ &\leq d^\ell \cdot \|T_i^{-1} T\|^\ell \cdot \max_{|\alpha| \leq \ell} |(\partial^\alpha h)(T_i^{-1} T \xi - T_i^{-1} b_i)| \\ &\quad (\text{since } \ell \leq N) \leq d^\ell \cdot \|T_i^{-1} T\|^\ell \cdot (h^\heartsuit \circ S_i^{-1} \circ T)(\xi). \end{aligned}$$

Now, set $g := \varphi_i \cdot \widehat{\gamma^{[j]}} \in C_c^\infty(\mathbb{R}^d)$ and apply Leibniz's rule to the product

$$g \circ T = (\widehat{\gamma^{[j]}} \circ T) \cdot (\varphi_i \circ T) = (\widehat{\gamma} \circ S_j^{-1} \circ T) \cdot (\varphi_i^\flat \circ S_i^{-1} \circ T),$$

to see using the binomial theorem that

$$\begin{aligned} |\partial_m^N (g \circ T)| &= \left| \sum_{\ell=0}^N \binom{N}{\ell} \cdot \partial_m^\ell (\widehat{\gamma} \circ S_j^{-1} \circ T) \cdot \partial_m^{N-\ell} (\varphi_i^\flat \circ S_i^{-1} \circ T) \right| \\ &\leq [\widehat{\gamma}^\heartsuit \circ S_j^{-1} \circ T] \cdot [(\varphi_i^\flat)^\heartsuit \circ S_i^{-1} \circ T] \cdot \sum_{\ell=0}^N \left[\binom{N}{\ell} \cdot d^N \cdot \|T_j^{-1} T\|^\ell \|T_i^{-1} T\|^{N-\ell} \right] \\ &= d^N \cdot (\|T_j^{-1} T\| + \|T_i^{-1} T\|)^N \cdot [\widehat{\gamma}^\heartsuit \circ S_j^{-1} \circ T] \cdot [(\varphi_i^\flat)^\heartsuit \circ S_i^{-1} \circ T]. \end{aligned}$$

Now, set $C_2 := \max_{|\alpha| \leq N} C^{(\alpha)}$, with $C^{(\alpha)}$ as in Assumption 6.1, equation (6.1). Because of $\text{supp } \varphi_i^\flat \subset Q'_i$, this yields $(\varphi_i^\flat)^\heartsuit \leq C_2 \cdot \mathbb{1}_{Q'_i}$ and thus

$$(\varphi_i^\flat)^\heartsuit \circ S_i^{-1} \circ T \leq C_2 \cdot \mathbb{1}_{T^{-1}(Q_i)} = C_2 \cdot \mathbb{1}_{Q_i} \circ T.$$

Hence,

$$\begin{aligned} |\partial_m^N(g \circ T)| &\leq d^N C_2 \cdot (\|T_j^{-1}T\| + \|T_i^{-1}T\|)^N \cdot [\hat{\gamma}^\heartsuit \circ S_j^{-1} \circ T] \cdot (\mathbf{1}_{Q_i} \circ T) \\ &= d^N C_2 \cdot (\|T_j^{-1}T\| + \|T_i^{-1}T\|)^N \cdot [(\hat{\gamma}^\heartsuit \circ S_j^{-1}) \cdot \mathbf{1}_{Q_i}] \circ T, \end{aligned}$$

as well as

$$|g \circ T| = |(\hat{\gamma} \circ S_j^{-1} \circ T) \cdot (\varphi_i^\flat \circ S_i^{-1} \circ T)| \leq C_2 \cdot [(\hat{\gamma}^\heartsuit \circ S_j^{-1}) \cdot \mathbf{1}_{Q_i}] \circ T.$$

By combining Lemma 6.3, equation (6.2) (with $g \circ T$ instead of g) with the preceding estimates, we arrive at

$$\begin{aligned} (1 + |x|)^N \cdot |[\mathcal{F}^{-1}(g \circ T)](x)| &\leq (1 + d)^N \cdot \left(|[\mathcal{F}^{-1}(g \circ T)](x)| + \sum_{m=1}^d |[\mathcal{F}^{-1}(\partial_m^N(g \circ T))](x)| \right) \\ &\leq (1 + d)^N \cdot \left(\|g \circ T\|_{L^1} + \sum_{m=1}^d \|\partial_m^N(g \circ T)\|_{L^1} \right) \\ &\leq (1 + d)^N C_2 \cdot \|[(\hat{\gamma}^\heartsuit \circ S_j^{-1}) \cdot \mathbf{1}_{Q_i}] \circ T\|_{L^1} \cdot \left(1 + \sum_{m=1}^d d^N \cdot (\|T_j^{-1}T\| + \|T_i^{-1}T\|)^N \right) \\ &\stackrel{(\dagger)}{\leq} d^{N+1} (1 + d)^N C_2 \cdot (1 + \|T_j^{-1}T\| + \|T_i^{-1}T\|)^N \cdot |\det T|^{-1} \cdot \|(\hat{\gamma}^\heartsuit \circ S_j^{-1}) \cdot \mathbf{1}_{Q_i}\|_{L^1} \\ &\leq (1 + d)^{1+2N} \cdot C_2 \cdot (1 + \|T_j^{-1}T\| + \|T_i^{-1}T\|)^N \cdot |\det T|^{-1} \cdot \|(\hat{\gamma}^\heartsuit \circ S_j^{-1}) \cdot \mathbf{1}_{Q_i}\|_{L^1} \\ &=: |\det T|^{-1} \cdot C_3^{(i,j,T)}, \end{aligned}$$

where the step marked with (\dagger) used that $1 + a^N \leq (1 + a)^N$ for $a \geq 0$, as can be seen by expanding the right-hand side using the binomial theorem.

Now, we choose $T = T_i$ and note $[\mathcal{F}^{-1}(g \circ T_i)](x) = |\det T_i|^{-1} \cdot (\mathcal{F}^{-1}g)(T_i^{-T}x)$, so that we have shown $|(\mathcal{F}^{-1}g)(T_i^{-T}x)| \leq C_3^{(i,j,T_i)} \cdot (1 + |x|)^{-N}$ and thus $|(\mathcal{F}^{-1}g)(y)| \leq C_3^{(i,j,T_i)} \cdot (1 + |T_i^T y|)^{-N}$ for all $y \in \mathbb{R}^d$. In conjunction with equation (1.11) and because of $v_0(y) \leq \Omega_1 \cdot (1 + |y|)^K$, we arrive at

$$\begin{aligned} v_0(y) \cdot |(\mathcal{F}^{-1}g)(y)| &\leq \Omega_1 \cdot (1 + |y|)^K \cdot |(\mathcal{F}^{-1}g)(y)| \\ &\stackrel{(\text{eq. (1.11)})}{\leq} \Omega_0^K \Omega_1 \cdot C_3^{(i,j,T_i)} \cdot (1 + |T_i^T y|)^{K-N}. \end{aligned}$$

By taking the L^p -quasi-norm of this estimate, we arrive at

$$\begin{aligned} \|\mathcal{F}^{-1}g\|_{L_{v_0}^p} &\leq \Omega_0^K \Omega_1 \cdot C_3^{(i,j,T_i)} \cdot \left\| (1 + |T_i^T \bullet|)^{K-N} \right\|_{L^p} \\ &= \Omega_0^K \Omega_1 \cdot C_3^{(i,j,T_i)} \cdot |\det T_i|^{-1/p} \cdot \left\| (1 + |\bullet|)^{-(N-K)} \right\|_{L^p} \\ &\stackrel{(\text{eq. (1.9)})}{\leq} \Omega_0^K \Omega_1 \cdot C_3^{(i,j,T_i)} \cdot |\det T_i|^{-1/p} \cdot \left(\frac{s_d}{\varepsilon} \right)^{1/p}, \end{aligned}$$

where the last step used our choice of $N = \left\lceil K + \frac{d+\varepsilon}{p} \right\rceil$.

This proves the claim, since

$$\begin{aligned} C_3^{(i,j,T_i)} &= (1 + d)^{1+2N} \cdot C_2 \cdot (2 + \|T_j^{-1}T_i\|)^N \cdot \|(\hat{\gamma}^\heartsuit \circ S_j^{-1}) \cdot \mathbf{1}_{Q_i}\|_{L^1} \\ &\leq (4d)^{1+2N} \cdot C_2 \cdot (1 + \|T_j^{-1}T_i\|)^N \cdot \int_{Q_i} \max_{|\alpha| \leq \left\lceil K + \frac{d+\varepsilon}{p} \right\rceil} |(\partial^\alpha \hat{\gamma})(S_j^{-1}\xi)| \, d\xi. \end{aligned} \quad \square$$

As a consequence of the preceding estimate, we see in particular that every regular \mathcal{Q} -BAPU is also a \mathcal{Q} - v_0 -BAPU, even for $v_0 \not\equiv 1$.

Corollary 6.5. *Every regular \mathcal{Q} -BAPU $\Phi = (\varphi_i)_{i \in I}$ is a \mathcal{Q} - v_0 -BAPU.*

In fact, there is some $\varrho \in C_c^\infty(\mathbb{R}^d)$, depending only on $\mathcal{Q} := \bigcup_{i \in I} \overline{Q'_i}$ (and thus only on \mathcal{Q}), such that

$$C_{\mathcal{Q}, \Phi, v_0, p} \leq \Omega_0^K \Omega_1 \cdot (4 \cdot d)^{1+2 \left\lceil K + \frac{d+\varepsilon}{p} \right\rceil} \cdot \left(\frac{s_d}{\varepsilon} \right)^{1/p} \cdot 2^{\left\lceil K + \frac{d+\varepsilon}{p} \right\rceil} \cdot \lambda_d(\mathcal{Q}) \cdot \max_{|\alpha| \leq \left\lceil K + \frac{d+\varepsilon}{p} \right\rceil} \|\partial^\alpha \varrho\|_{\sup} \cdot \max_{|\alpha| \leq \left\lceil K + \frac{d+\varepsilon}{p} \right\rceil} C^{(\alpha)},$$

where $\varepsilon > 0$ can be chosen arbitrarily. ◀

Proof. The set $Q \subset \mathbb{R}^d$ is compact, so that there is some $\gamma \in \mathcal{S}(\mathbb{R}^d)$ satisfying $\widehat{\gamma} \in C_c^\infty(\mathbb{R}^d)$ and $\gamma \equiv 1$ on Q . In the notation of Lemma 6.4, this entails $\widehat{\gamma[\mathbb{I}]} = \widehat{\gamma} \circ S_j^{-1} \equiv 1$ on $S_j Q \supset S_j \overline{Q'_j} = \overline{Q_j}$. But because of $\varphi_j \equiv 0$ outside of $\overline{Q_j}$, this implies $\widehat{\gamma[\mathbb{I}]} \cdot \varphi_j = \varphi_j$, so that Lemma 6.4 yields (with C_0 as in that lemma) that

$$\begin{aligned} \|\mathcal{F}^{-1}\varphi_j\|_{L_{v_0}^p} &= \left\| \mathcal{F}^{-1} \left(\widehat{\gamma[\mathbb{I}]} \cdot \varphi_j \right) \right\|_{L_{v_0}^p} \\ &\leq C_0 \cdot (1 + \|T_j^{-1}T_j\|)^{\lceil K + \frac{d+\varepsilon}{p} \rceil} \cdot |\det T_j|^{1-\frac{1}{p}} \cdot \int_{Q'_j} \max_{|\alpha| \leq \lceil K + \frac{d+\varepsilon}{p} \rceil} |(\partial^\alpha \widehat{\gamma})(S_j^{-1}S_j\xi)| \, d\xi \\ &\leq C_0 \cdot 2^{\lceil K + \frac{d+\varepsilon}{p} \rceil} \cdot |\det T_j|^{1-\frac{1}{p}} \cdot \lambda_d(Q'_j) \cdot \max_{|\alpha| \leq \lceil K + \frac{d+\varepsilon}{p} \rceil} \|\partial^\alpha \widehat{\gamma}\|_{\sup} \\ &\leq C_0 \cdot 2^{\lceil K + \frac{d+\varepsilon}{p} \rceil} \cdot \lambda_d(Q) \cdot \max_{|\alpha| \leq \lceil K + \frac{d+\varepsilon}{p} \rceil} \|\partial^\alpha \widehat{\gamma}\|_{\sup} \cdot |\det T_j|^{1-\frac{1}{p}} \\ &=: C \cdot |\det T_j|^{1-\frac{1}{p}}, \end{aligned}$$

where $C > 0$ is independent of $j \in I$. Recalling the definition of a \mathcal{Q} - v_0 -BAPU from Subsection 1.3, this yields the claim, with $\varrho := \widehat{\gamma}$. \square

Using Schur's test as well as the estimates given in Lemma 6.4, we can now derive simplified sufficient criteria which ensure that a given family $\Gamma = (\gamma_i)_{i \in I}$ of prototypes indeed generates a Banach frame (as in Theorem 4.7) or an atomic decomposition (as in Theorem 5.6). We start with a simplified criterion for Banach frames.

Corollary 6.6. *Assume that $(\varphi_i)_{i \in I}$ satisfies Assumption 6.1. Then, for each $p, q \in (0, \infty]$, there are*

$$N \in \mathbb{N}, \quad \sigma > 0, \quad \text{and} \quad \tau > 0$$

with the following property: If the family $\Gamma = (\gamma_i)_{i \in I}$ satisfies the following:

- (1) *We have $\gamma_i \in L_{(1+|\bullet|)^K}^1(\mathbb{R}^d)$ and $\widehat{\gamma}_i \in C^\infty(\mathbb{R}^d)$ for all $i \in I$, where all partial derivatives of $\widehat{\gamma}_i$ are polynomially bounded.*
- (2) *We have $\gamma_i \in C^1(\mathbb{R}^d)$ and $\partial_\ell \gamma_i \in L_{v_0}^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ for all $\ell \in \underline{d}$ and $i \in I$.*
- (3) *The family $\Gamma = (\gamma_i)_{i \in I}$ satisfies Assumption 3.6.*
- (4) *We have*

$$C_1 := \sup_{i \in I} \sum_{j \in I} M_{j,i} < \infty \quad \text{and} \quad C_2 := \sup_{j \in I} \sum_{i \in I} M_{j,i} < \infty$$

with

$$M_{j,i} := \left(\frac{w_j}{w_i} \right)^\tau \cdot (1 + \|T_j^{-1}T_i\|)^\sigma \cdot \max_{|\beta| \leq 1} \left(|\det T_i|^{-1} \cdot \int_{Q_i} \max_{|\alpha| \leq N} |(\partial^\alpha \widehat{\partial^\beta \gamma_j})(S_j^{-1}\xi)| \, d\xi \right)^\tau.$$

Then Γ fulfills Assumptions 4.1 and 3.6 and thus all assumptions of Theorem 4.7.

In fact, the following choices are possible, for an arbitrary $\varepsilon > 0$:

$$N = \left\lceil K + \frac{d + \varepsilon}{\min\{1, p\}} \right\rceil,$$

$$\tau = \min\{1, p, q\} = \begin{cases} \min\{1, q\}, & \text{if } p \in [1, \infty], \\ \min\{q, p\}, & \text{if } p \in (0, 1), \end{cases}$$

$$\sigma = \tau \cdot \left(\frac{d}{\min\{1, p\}} + K + \left\lceil K + \frac{d + \varepsilon}{\min\{1, p\}} \right\rceil \right) = \begin{cases} \min\{1, q\} \cdot (d + K + \lceil K + d + \varepsilon \rceil), & \text{if } p \in [1, \infty], \\ \min\{p, q\} \cdot \left(\frac{d}{p} + K + \left\lceil K + \frac{d + \varepsilon}{p} \right\rceil \right), & \text{if } p \in (0, 1). \end{cases}$$

With these choices, we even have $\|\vec{A}\|^{\max\{1, \frac{1}{p}\}} \leq C \cdot (C_1^{1/\tau} + C_2^{1/\tau})$ and $\|\vec{B}\|^{\max\{1, \frac{1}{p}\}} \leq C \cdot (C_1^{1/\tau} + C_2^{1/\tau})$ for

$$C := \Omega_0^K \Omega_1 \cdot d^{1/\min\{1, p\}} \cdot (4 \cdot d)^{1+2\lceil K + \frac{d+\varepsilon}{\min\{1, p\}} \rceil} \cdot \left(\frac{s_d}{\varepsilon} \right)^{1/\min\{1, p\}} \cdot \max_{|\alpha| \leq \lceil K + \frac{d+\varepsilon}{\min\{1, p\}} \rceil} C^{(\alpha)}. \quad \blacktriangleleft$$

Remark. As usual, the most important special case is when $\gamma_i = \gamma$ is independent of $i \in I$. In this case, validity of Assumption 3.6 can be verified easily using Lemma 3.7. The same lemma is also highly helpful if $\{\gamma_i \mid i \in I\}$ is finite, i.e., if only a finite number of different prototypes is used. \blacklozenge

Proof. Since our assumptions clearly include those of Assumption 3.6, we only need to verify Assumption 4.1. This means the following:

- We have $\gamma_i \in C^1(\mathbb{R}^d)$ and the gradient $\phi_i := \nabla \gamma_i$ is bounded and satisfies $\phi_i \in L_{v_0}^1(\mathbb{R}^d; \mathbb{C}^d)$, as well as $\widehat{\phi_i} \in C^\infty(\mathbb{R}^d; \mathbb{C}^d)$. All of these properties except the last are included in our assumptions. But standard properties of the Fourier transform show $\widehat{\partial_\ell \gamma_i}(\xi) = 2\pi i \xi_\ell \cdot \widehat{\gamma_i}(\xi)$ for $\xi \in \mathbb{R}^d$, so that $\widehat{\partial_\ell \gamma_i} \in C^\infty(\mathbb{R}^d)$, since $\widehat{\gamma_i} \in C^\infty(\mathbb{R}^d)$.
- Assumption 3.1 is satisfied. For this, it remains—in view of our present assumptions—to check that the operator $\vec{A} : \ell_{w^{\min\{1,p\}}}^r(I) \rightarrow \ell_{w^{\min\{1,p\}}}^r(I)$ is bounded, where $r := \max\left\{q, \frac{q}{p}\right\}$ and $A = (A_{j,i})_{j,i \in I}$ is given by

$$A_{j,i} := \begin{cases} \left\| \mathcal{F}^{-1} \left(\varphi_i \cdot \widehat{\gamma^{(j)}} \right) \right\|_{L_{v_0}^1}, & \text{if } p \in [1, \infty], \\ (1 + \|T_j^{-1} T_i\|)^d \cdot |\det T_i|^{1-p} \cdot \left\| \mathcal{F}^{-1} \left(\varphi_i \cdot \widehat{\gamma^{(j)}} \right) \right\|_{L_{v_0}^p}^p, & \text{if } p \in (0, 1). \end{cases}$$

- The infinite matrix $B = (B_{j,i})_{j,i \in I}$ defines a bounded linear operator $\vec{B} : \ell_{w^{\min\{1,p\}}}^r(I) \rightarrow \ell_{w^{\min\{1,p\}}}^r(I)$, where $r = \max\left\{q, \frac{q}{p}\right\}$ as above, $\phi_i = \nabla \gamma_i$ for $i \in I$ and

$$B_{j,i} := \begin{cases} (1 + \|T_j^{-1} T_i\|)^{K+d} \cdot \left\| \mathcal{F}^{-1} \left(\varphi_i \cdot \widehat{\phi^{(j)}} \right) \right\|_{L_{v_0}^1}, & \text{if } p \in [1, \infty], \\ (1 + \|T_j^{-1} T_i\|)^{pK+d} \cdot |\det T_i|^{1-p} \cdot \left\| \mathcal{F}^{-1} \left(\varphi_i \cdot \widehat{\phi^{(j)}} \right) \right\|_{L_{v_0}^p}^p, & \text{if } p \in (0, 1). \end{cases}$$

Here, $\phi^{(j)}$ is defined as in equation (3.1), with γ_j replaced by ϕ_j .

Hence, in the following, we verify boundedness of \vec{A} and \vec{B} .

We first make the auxiliary observation that a matrix operator $\vec{C} : \ell_v^q(I) \rightarrow \ell_v^q(I)$ is bounded if and only if the operator $\vec{C}_v : \ell^q(I) \rightarrow \ell^q(I)$ is bounded, where

$$(C_v)_{j,i} = \frac{v_j}{v_i} \cdot C_{j,i}.$$

This simply comes from the fact that $m_v : \ell_v^q(I) \rightarrow \ell^q(I)$, $(x_i)_{i \in I} \mapsto (v_i x_i)_{i \in I}$ is an isometric isomorphism and that

$$\left[\vec{C} : \ell_v^q(I) \rightarrow \ell_v^q(I) \right] = m_v^{-1} \circ \left(m_v \circ \vec{C} \circ m_v^{-1} \right) \circ m_v,$$

where a direct calculation shows $m_v \circ \vec{C} \circ m_v^{-1} = \vec{C}_v$. Since m_v is isometric, we also get $\|\vec{C}_v\| = \|\vec{C}\|$.

Now, let us first consider the case $p \in [1, \infty]$. Here, we want to have $\vec{A} : \ell_w^q(I) \rightarrow \ell_w^q(I)$ and likewise for \vec{B} . Recall $\gamma^{(j)} = \gamma_j^{[j]}$ in the notation of Lemma 6.4. Hence, an application of that lemma (with $p = 1$) yields, with C as in the statement of the present corollary,

$$\begin{aligned} (A_w)_{j,i} &= \frac{w_j}{w_i} \cdot A_{j,i} \leq \frac{C}{d} \cdot \frac{w_j}{w_i} \cdot [1 + \|T_j^{-1} T_i\|]^{K+d+\varepsilon} \cdot |\det T_i|^{-1} \cdot \int_{Q_i} \max_{|\alpha| \leq \lceil K+d+\varepsilon \rceil} |(\partial^\alpha \widehat{\gamma_j})(S_j^{-1} \xi)| \, d\xi \\ &\leq C \cdot \frac{w_j}{w_i} \cdot [1 + \|T_j^{-1} T_i\|]^{K+d+\lceil K+d+\varepsilon \rceil} \cdot |\det T_i|^{-1} \cdot \max_{|\beta| \leq 1} \int_{Q_i} \max_{|\alpha| \leq N} \left| (\partial^\alpha \widehat{\partial^\beta \gamma_j})(S_j^{-1} \xi) \right| \, d\xi \\ &\leq C \cdot M_{j,i}^{1/\min\{1,q\}}. \end{aligned}$$

Likewise, using $|\phi_j| = |\widehat{\nabla \gamma_j}| \leq \sum_{\ell=1}^d |\widehat{\partial_\ell \gamma_j}|$, we get

$$(B_w)_{j,i} = \frac{w_j}{w_i} \cdot B_{j,i} \leq d \cdot \frac{w_j}{w_i} \cdot (1 + \|T_j^{-1} T_i\|)^{K+d} \cdot \max_{|\beta|=1} \left\| \mathcal{F}^{-1} \left(\varphi_i \cdot \widehat{(\partial^\beta \gamma_j)^{[j]}} \right) \right\|_{L_{v_0}^1}$$

$$\begin{aligned} (\text{Lemma 6.4 with } \partial^\beta \gamma_j \text{ instead of } \gamma) &\leq C \cdot \frac{w_j}{w_i} \cdot (1 + \|T_j^{-1} T_i\|)^{K+d+\lceil K+d+\varepsilon \rceil} \cdot \max_{|\beta|=1} \left[|\det T_i|^{-1} \int_{Q_i} \max_{|\alpha| \leq N} \left| (\partial^\alpha \widehat{\partial^\beta \gamma_j})(S_j^{-1} \xi) \right| \, d\xi \right] \\ &\leq C \cdot M_{j,i}^{1/\min\{1,q\}}. \end{aligned}$$

But Lemma 6.2 shows that $\vec{A}_w : \ell^q(I) \rightarrow \ell^q(I)$ is bounded as soon as we have $K_1 := \sup_{i \in I} \sum_{j \in I} (A_w)_{j,i}^{\min\{1,q\}} < \infty$ and $K_2 := \sup_{j \in I} \sum_{i \in I} (A_w)_{j,i}^{\min\{1,q\}} < \infty$. Further, that lemma also shows $\|\vec{A}_w\| \leq \max\left\{K_1^{1/\min\{1,q\}}, K_2^{1/\min\{1,q\}}\right\}$.

Since we have by assumption that $C_1 = \sup_{i \in I} \sum_{j \in I} M_{j,i} < \infty$ and $C_2 = \sup_{j \in I} \sum_{i \in I} M_{j,i} < \infty$, we get $K_1^{1/\min\{1,q\}} \leq C \cdot C_1^{1/\min\{1,q\}} = C \cdot C_1^{1/\tau}$ and likewise $K_2^{1/\min\{1,q\}} \leq C \cdot C_2^{1/\min\{1,q\}} = C \cdot C_2^{1/\tau}$, so that $\|\vec{A}_w\| \leq C \cdot \max\{C_1^{1/\tau}, C_2^{1/\tau}\}$. The same arguments show that $\vec{B}_w : \ell^q(I) \rightarrow \ell^q(I)$ is bounded and satisfies $\|\vec{B}_w\| \leq C \cdot \max\{C_1^{1/\tau}, C_2^{1/\tau}\}$. In view of the auxiliary observation from above, this completes the proof in case of $p \in [1, \infty]$.

Now, let $p \in (0, 1)$. In this case, we want to have $\vec{A} : \ell_{w^p}^{q/p}(I) \rightarrow \ell_{w^p}^{q/p}(I)$ and likewise for \vec{B} . But Lemma 6.4 yields, because of $\gamma^{(j)} = \gamma_j^{[j]}$,

$$\begin{aligned} (A_{w^p})_{j,i} &= \left(\frac{w_j}{w_i}\right)^p \cdot A_{j,i} \\ &= \left(\frac{w_j}{w_i}\right)^p \cdot (1 + \|T_j^{-1}T_i\|)^d \cdot |\det T_i|^{1-p} \cdot \left\| \mathcal{F}^{-1} \left(\varphi_i \cdot \widehat{\gamma^{(j)}} \right) \right\|_{L_{v_0}^p}^p \\ &\leq (C/d)^p \cdot \left(\frac{w_j}{w_i}\right)^p (1 + \|T_j^{-1}T_i\|)^{d+p \lceil K + \frac{d+\varepsilon}{p} \rceil} |\det T_i|^{1-p} |\det T_i|^{-1} \left(\int_{Q_i} \max_{|\alpha| \leq \lceil K + \frac{d+\varepsilon}{p} \rceil} |(\partial^\alpha \widehat{\gamma_j})(S_j^{-1}\xi)| \, d\xi \right)^p \\ &= (C/d)^p \cdot \left(\frac{w_j}{w_i}\right)^p (1 + \|T_j^{-1}T_i\|)^{d+p \lceil K + \frac{d+\varepsilon}{p} \rceil} \left(|\det T_i|^{-1} \int_{Q_i} \max_{|\alpha| \leq N} |(\partial^\alpha \widehat{\gamma_j})(S_j^{-1}\xi)| \, d\xi \right)^p \\ &\leq (C/d)^p \cdot M_{j,i}^{1/\min\{1, \frac{q}{p}\}}, \end{aligned}$$

where the last step used

$$\min \left\{ 1, \frac{q}{p} \right\} \cdot \left(d + p \left\lceil K + \frac{d+\varepsilon}{p} \right\rceil \right) = \min \{p, q\} \cdot \left(\frac{d}{p} + \left\lceil K + \frac{d+\varepsilon}{p} \right\rceil \right) \leq \sigma.$$

Furthermore, since $p \in (0, 1)$, we have $\left(\sum_{\ell=1}^d a_\ell \right)^p \leq \sum_{\ell=1}^d a_\ell^p$ for $a_1, \dots, a_d \geq 0$, so that the L^p -norm of a vector valued (measurable) function $f : \mathbb{R}^d \rightarrow \mathbb{C}^d$ can be estimated as follows:

$$\begin{aligned} \|(f_1, \dots, f_d)\|_{L^p}^p &= \int_{\mathbb{R}^d} |(f_1, \dots, f_d)(x)|^p \, dx \leq \int_{\mathbb{R}^d} \left(\sum_{\ell=1}^d |f_\ell(x)| \right)^p \, dx \\ &\leq \int_{\mathbb{R}^d} \sum_{\ell=1}^d |f_\ell(x)|^p \, dx = \sum_{\ell=1}^d \|f_\ell\|_{L^p}^p \leq d \cdot \max_{\ell \in \underline{d}} \|f_\ell\|_{L^p}^p. \end{aligned}$$

Consequently,

$$\begin{aligned} (B_{w^p})_{j,i} &= \left(\frac{w_j}{w_i}\right)^p \cdot B_{j,i} \\ &= \left(\frac{w_j}{w_i}\right)^p \cdot (1 + \|T_j^{-1}T_i\|)^{pK+d} \cdot |\det T_i|^{1-p} \cdot \left\| \mathcal{F}^{-1} \left(\varphi_i \cdot \widehat{\phi^{(j)}} \right) \right\|_{L_{v_0}^p}^p \\ &\quad (\text{since } \phi_j = \nabla \gamma_j) \leq d \cdot \left(\frac{w_j}{w_i}\right)^p \cdot (1 + \|T_j^{-1}T_i\|)^{pK+d} \cdot |\det T_i|^{1-p} \cdot \max_{|\beta|=1} \left\| \mathcal{F}^{-1} \left(\varphi_i \cdot \widehat{(\partial^\beta \gamma_j)^{[j]}} \right) \right\|_{L_{v_0}^p}^p \\ &\quad (\text{Lem. 6.4 / w } \partial^\beta \gamma_j \text{ inst. of } \gamma_j) \leq d \cdot \left(C/d^{\frac{1}{p}}\right)^p \cdot \left(\frac{w_j}{w_i}\right)^p (1 + \|T_j^{-1}T_i\|)^{pK+d} \cdot |\det T_i|^{1-p} \\ &\quad \cdot \max_{|\beta| \leq 1} \left[(1 + \|T_j^{-1}T_i\|)^{\lceil K + \frac{d+\varepsilon}{p} \rceil} |\det T_i|^{-\frac{1}{p}} \int_{Q_i} \max_{|\alpha| \leq \lceil K + \frac{d+\varepsilon}{p} \rceil} |(\partial^\alpha \widehat{\partial^\beta \gamma_j})(S_j^{-1}\xi)| \, d\xi \right]^p \\ &= C^p \cdot \left(\frac{w_j}{w_i}\right)^p (1 + \|T_j^{-1}T_i\|)^{p(\frac{d}{p} + K + \lceil K + \frac{d+\varepsilon}{p} \rceil)} \\ &\quad \cdot \max_{|\beta| \leq 1} \left[|\det T_i|^{-1} \int_{Q_i} \max_{|\alpha| \leq N} |(\partial^\alpha \widehat{\partial^\beta \gamma_j})(S_j^{-1}\xi)| \, d\xi \right]^p \\ &= C^p \cdot M_{j,i}^{1/\min\{1, \frac{q}{p}\}}. \end{aligned}$$

Now, the remainder of the proof is similar to the case $p \in [1, \infty]$: Lemma 6.2 shows that $\overrightarrow{A_{w^p}} : \ell^{q/p}(I) \rightarrow \ell^{q/p}(I)$ is bounded as soon as we have $K_3 := \sup_{i \in I} \sum_{j \in I} (A_{w^p})_{j,i}^{\min\{1, \frac{q}{p}\}} < \infty$ and $K_4 := \sup_{j \in I} \sum_{i \in I} (A_{w^p})_{j,i}^{\min\{1, \frac{q}{p}\}} < \infty$. Further, that lemma also shows $\|\overrightarrow{A_{w^p}}\| \leq \max\left\{K_3^{1/\min\{1, q/p\}}, K_4^{1/\min\{1, q/p\}}\right\}$. Since we have by assumption that $C_1 = \sup_{i \in I} \sum_{j \in I} M_{j,i} < \infty$ and $C_2 = \sup_{j \in I} \sum_{i \in I} M_{j,i} < \infty$, we get

$$\begin{aligned} \|\overrightarrow{A_{w^p}}\|^{\max\{1, \frac{1}{p}\}} &= \|\overrightarrow{A_{w^p}}\|^{1/p} \\ &\leq \max\left\{K_3^{\frac{1}{p} \cdot \frac{1}{\min\{1, q/p\}}}, K_4^{\frac{1}{p} \cdot \frac{1}{\min\{1, q/p\}}}\right\} \\ &= \max\left\{K_3^{1/\min\{p, q\}}, K_4^{1/\min\{p, q\}}\right\} \\ &\leq \max\left\{\left(\sup_{i \in I} \sum_{j \in I} \left[C^{p \cdot \min\{1, \frac{q}{p}\}} \cdot M_{j,i}\right]\right)^{1/\min\{p, q\}}, \left(\sup_{j \in I} \sum_{i \in I} \left[C^{p \cdot \min\{1, \frac{q}{p}\}} \cdot M_{j,i}\right]\right)^{1/\min\{p, q\}}\right\} \\ &\leq \max\left\{C \cdot C_1^{1/\tau}, C \cdot C_2^{1/\tau}\right\} = C \cdot \max\left\{C_1^{1/\tau}, C_2^{1/\tau}\right\}. \end{aligned}$$

Exactly the same arguments also yield $\|\overrightarrow{B_{w^p}}\|^{\max\{1, \frac{1}{p}\}} \leq C \cdot \max\left\{C_1^{1/\tau}, C_2^{1/\tau}\right\}$. \square

Our next result yields simplified criteria for the application of Theorem 5.6, which yields atomic decompositions for $\mathcal{D}(\mathcal{Q}, L_v^p, \ell_w^q)$.

Corollary 6.7. *Assume that $(\varphi_i)_{i \in I}$ satisfies Assumption 6.1. Then, for each $p, q \in (0, \infty]$, there are*

$$N \in \mathbb{N}, \quad \sigma > 0, \quad \vartheta \geq 0 \quad \text{and} \quad \tau > 0$$

with the following property: If the families $\Gamma = (\gamma_i)_{i \in I}$ and $\Gamma_\ell = (\gamma_{i,\ell})_{i \in I}$ (with $\ell \in \{1, 2\}$) satisfy the following properties:

- (1) All $\gamma_i, \gamma_{i,1}, \gamma_{i,2}$ are measurable functions $\mathbb{R}^d \rightarrow \mathbb{C}$,
- (2) We have $\gamma_{i,1} \in L^1_{(1+|\bullet|)^K}(\mathbb{R}^d)$ for all $i \in I$.
- (3) We have $\gamma_{i,2} \in C^1(\mathbb{R}^d)$ for all $i \in I$.
- (4) With $K_0 := K + \frac{d}{\min\{1, p\}} + 1$, we have

$$\Omega_4^{(p,K)} := \sup_{i \in I} \|\gamma_{i,2}\|_{K_0} + \sup_{i \in I} \|\nabla \gamma_{i,2}\|_{K_0} < \infty,$$

where $\|f\|_{K_0} := \sup_{x \in \mathbb{R}^d} (1 + |x|)^{K_0} |f(x)|$.

- (5) We have $\|\gamma_i\|_{K_0} < \infty$ for all $i \in I$.
- (6) We have $\gamma_i = \gamma_{i,1} * \gamma_{i,2}$ for all $i \in I$.
- (7) We have $\widehat{\gamma_{i,1}}, \widehat{\gamma_{i,2}} \in C^\infty(\mathbb{R}^d)$ and all partial derivatives of $\widehat{\gamma_{i,1}}, \widehat{\gamma_{i,2}}$ are polynomially bounded for all $i \in I$.
- (8) $\Gamma = (\gamma_i)_{i \in I}$ satisfies Assumption 3.6.
- (9) We have

$$K_1 := \sup_{i \in I} \sum_{j \in I} N_{i,j} < \infty \quad \text{and} \quad K_2 := \sup_{j \in I} \sum_{i \in I} N_{i,j} < \infty$$

with

$$N_{i,j} := \left(\frac{w_i}{w_j} \cdot (|\det T_j| / |\det T_i|)^\vartheta\right)^\tau \cdot (1 + \|T_j^{-1} T_i\|)^\sigma \cdot \left(|\det T_i|^{-1} \cdot \int_{Q_i} \max_{|\alpha| \leq N} |(\partial^\alpha \widehat{\gamma_{j,1}})(S_j^{-1} \xi)| \, d\xi\right)^\tau.$$

Then the families $\Gamma, \Gamma_1, \Gamma_2$ fulfill Assumption 5.1 and the family Γ satisfies Assumption 3.6, so that Theorem 5.6 is applicable to Γ .

In fact, the following choices are possible, for an arbitrary $\varepsilon > 0$:

$$\begin{aligned} N &= \left\lceil K + \frac{d + \varepsilon}{\min\{1, p\}} \right\rceil, \\ \tau &= \min\{1, p, q\} = \begin{cases} \min\{1, q\}, & \text{if } p \in [1, \infty], \\ \min\{p, q\}, & \text{if } p \in (0, 1), \end{cases} \\ \sigma &= \begin{cases} \min\{1, q\} \cdot \lceil K + d + \varepsilon \rceil, & \text{if } p \in [1, \infty], \\ \min\{p, q\} \cdot \left(\frac{d}{p} + K + \left\lceil K + \frac{d + \varepsilon}{p} \right\rceil\right), & \text{if } p \in (0, 1), \end{cases} \\ \vartheta &= \begin{cases} 0, & \text{if } p \in [1, \infty], \\ \frac{1}{p} - 1, & \text{if } p \in (0, 1). \end{cases} \end{aligned}$$

With these choices, we even have $\|\vec{C}\|_{\ell_{w^{\min\{1, p\}}}^{\max\{1, \frac{1}{p}\}}(I)} \leq \Omega \cdot (K_1^{1/\tau} + K_2^{1/\tau})$, where $\vec{C} : \ell_{w^{\min\{1, p\}}}^{\max\{q, q/p\}}(I) \rightarrow \ell_{w^{\min\{1, p\}}}^{\max\{q, q/p\}}(I)$ is defined as in Assumption 5.1 and where

$$\Omega := \Omega_0^K \Omega_1 \cdot (4 \cdot d)^{1+2\lceil K + \frac{d + \varepsilon}{\min\{1, p\}} \rceil} \cdot \left(\frac{sd}{\varepsilon}\right)^{1/\min\{1, p\}} \cdot \max_{|\alpha| \leq N} C^{(\alpha)} \quad \blacktriangleleft$$

Proof. First, our assumptions clearly imply that Assumption 3.6 is satisfied. Hence, we only need to verify Assumption 5.1, which means the following:

- We have $\gamma_{i,1}, \gamma_{i,2} \in L_{(1+|\bullet|)^K}^1(\mathbb{R}^d)$ for all $i \in I$. For $\gamma_{i,1}$, this is part of our assumptions. But for $\gamma_{i,2}$, we have $\|\gamma_{i,2}\|_{K_0} < \infty$, which implies $\|(1 + |\bullet|)^K \cdot \gamma_{i,2}\|_{K_0 - K} < \infty$. Because of $K_0 - K = \frac{d}{\min\{1, p\}} + 1 \geq d + 1$, this easily implies $\gamma_{i,2} \in L_{(1+|\bullet|)^K}^1(\mathbb{R}^d)$; see also the remark after Assumption 5.1.
- We have $\gamma_i = \gamma_{i,1} * \gamma_{i,2}$ for all $i \in I$, which is part of our assumptions.
- We have $\widehat{\gamma_{i,1}}, \widehat{\gamma_{i,2}} \in C^\infty(\mathbb{R}^d)$, with all partial derivatives of $\widehat{\gamma_{i,1}}, \widehat{\gamma_{i,2}}$ being polynomially bounded. Again, this is part of our assumptions.
- We have $\gamma_{i,2} \in C^1(\mathbb{R}^d)$ with $\nabla \gamma_{i,2} \in L_{v_0}^1(\mathbb{R}^d)$. The first of these properties is part of our assumptions and the second property follows easily from $\|\nabla \gamma_{i,2}\|_{K_0} \leq \Omega_4^{(p,K)} < \infty$, cf. the remark after Assumption 5.1.
- We have $\Omega_4^{(p,K)} < \infty$, where the constant $\Omega_4^{(p,K)}$ is defined as in the present corollary. Hence, this prerequisite is part of our assumptions.
- We have $\|\gamma_i\|_{K_0} < \infty$ for all $i \in I$, which is again part of our assumptions.
- The operator $\vec{C} : \ell_{w^{\min\{1, p\}}}^r(I) \rightarrow \ell_{w^{\min\{1, p\}}}^r(I)$ is bounded, where $r = \max\{q, \frac{q}{p}\}$ and where

$$C_{i,j} := \begin{cases} \left\| \mathcal{F}^{-1} \left(\varphi_i \cdot \widehat{\gamma_1^{(j)}} \right) \right\|_{L_{v_0}^1}, & \text{if } p \in [1, \infty], \\ (1 + \|T_j^{-1} T_i\|)^{pK+d} \cdot |\det T_j|^{1-p} \cdot \left\| \mathcal{F}^{-1} \left(\varphi_i \cdot \widehat{\gamma_1^{(j)}} \right) \right\|_{L_{v_0}^p}^p, & \text{if } p \in (0, 1). \end{cases}$$

Thus, in the remainder of the proof, we only need to verify boundedness of \vec{C} . First of all, we recall from the proof of Corollary 6.6 that we have $\|\vec{C}\| = \left\| \overrightarrow{C_{w^{\min\{1, p\}}}} \right\|_{\ell^r \rightarrow \ell^r}$, where

$$(C_{w^{\min\{1, p\}}})_{i,j} := \frac{w_i}{w_j} \cdot C_{i,j}.$$

To prove boundedness of $\overrightarrow{C_{w^{\min\{1, p\}}}}$, we distinguish the cases $p \in [1, \infty]$ and $p \in (0, 1)$.

For $p \in [1, \infty]$, we want to have $\vec{C}_w : \ell^q(I) \rightarrow \ell^q(I)$. But Lemma 6.4 (with $p = 1$ and with $\gamma_{j,1}$ instead of γ) yields because of $\gamma_1^{(j)} = \gamma_{j,1}^{[j]}$ that

$$\begin{aligned} (C_w)_{i,j} &= \frac{w_i}{w_j} \cdot C_{i,j} = \frac{w_i}{w_j} \cdot \left\| \mathcal{F}^{-1} \left(\varphi_i \cdot \widehat{\gamma_1^{(j)}} \right) \right\|_{L_{v_0}^1} \\ &\leq \Omega \cdot \frac{w_i}{w_j} \cdot (1 + \|T_j^{-1} T_i\|)^{\lceil K+d+\varepsilon \rceil} \cdot |\det T_i|^{-1} \cdot \int_{Q_i} \max_{|\alpha| \leq N} |(\partial^\alpha \widehat{\gamma_{j,1}})(S_j^{-1} \eta)| \, d\eta \\ &= \Omega \cdot N_{i,j}^{1/\min\{1, q\}}. \end{aligned}$$

Finally, Lemma 6.2 shows that

$$\begin{aligned} \|\overrightarrow{C_w}\|_{\ell^q \rightarrow \ell^q} &\leq \max \left\{ \left(\sup_{i \in I} \sum_{j \in I} |(C_w)_{i,j}|^{\min\{1,q\}} \right)^{1/\min\{1,q\}}, \left(\sup_{j \in I} \sum_{i \in I} |(C_w)_{i,j}|^{\min\{1,q\}} \right)^{1/\min\{1,q\}} \right\} \\ &\leq \Omega \cdot \max \left\{ \left(\sup_{i \in I} \sum_{j \in I} N_{i,j} \right)^{1/\min\{1,q\}}, \left(\sup_{j \in I} \sum_{i \in I} N_{i,j} \right)^{1/\min\{1,q\}} \right\} \\ &= \Omega \cdot \max \{ K_1^{1/\tau}, K_2^{1/\tau} \}, \end{aligned}$$

as desired.

In case of $p \in (0, 1)$, we want to have $\overrightarrow{C_{w^p}} : \ell^{q/p}(I) \rightarrow \ell^{q/p}(I)$. But Lemma 6.4 (with $\gamma_{j,1}$ instead of γ) yields

$$\begin{aligned} (C_{w^p})_{i,j} &= \left(\frac{w_i}{w_j} \right)^p \cdot C_{i,j} = \left(\frac{w_i}{w_j} \right)^p \cdot (1 + \|T_j^{-1} T_i\|)^{pK+d} \cdot |\det T_j|^{1-p} \cdot \left\| \mathcal{F}^{-1} \left(\varphi_i \cdot \widehat{\gamma_1^{(j)}} \right) \right\|_{L_{v_0}^p}^p \\ &\leq \Omega^p \cdot \left(\frac{w_i}{w_j} \right)^p (1 + \|T_j^{-1} T_i\|)^{p(\frac{d}{p} + K + \lceil K + \frac{d+\varepsilon}{p} \rceil)} |\det T_j|^{1-p} |\det T_i|^{-1} \left(\int_{Q_i} \max_{|\alpha| \leq N} |(\partial^\alpha \widehat{\gamma_{j,1}})(S_j^{-1} \eta)| \, d\eta \right)^p \\ &= \Omega^p \cdot \left(\frac{w_i}{w_j} \cdot \left(\frac{|\det T_j|}{|\det T_i|} \right)^{\frac{1}{p}-1} (1 + \|T_j^{-1} T_i\|)^{\frac{d}{p} + K + \lceil K + \frac{d+\varepsilon}{p} \rceil} |\det T_i|^{-1} \int_{Q_i} \max_{|\alpha| \leq N} |(\partial^\alpha \widehat{\gamma_{j,1}})(S_j^{-1} \eta)| \, d\eta \right)^p \\ &\leq \Omega^p \cdot N_{i,j}^{1/\min\{1, \frac{q}{p}\}}. \end{aligned}$$

Finally, Lemma 6.2 shows that

$$\begin{aligned} \|\overrightarrow{C_{w^p}}\|_{\ell^{q/p} \rightarrow \ell^{q/p}}^{1/p} &\leq \left[\max \left\{ \left(\sup_{i \in I} \sum_{j \in I} (C_{w^p})_{i,j}^{\min\{1, \frac{q}{p}\}} \right)^{1/\min\{1, \frac{q}{p}\}}, \left(\sup_{j \in I} \sum_{i \in I} (C_{w^p})_{i,j}^{\min\{1, \frac{q}{p}\}} \right)^{1/\min\{1, \frac{q}{p}\}} \right\} \right]^{1/p} \\ &\leq \Omega \cdot \max \left\{ \left(\sup_{i \in I} \sum_{j \in I} N_{i,j} \right)^{1/\min\{p,q\}}, \left(\sup_{j \in I} \sum_{i \in I} N_{i,j} \right)^{1/\min\{p,q\}} \right\} \\ &= \Omega \cdot \max \{ K_1^{1/\tau}, K_2^{1/\tau} \}, \end{aligned}$$

as desired. \square

One remaining limitation of Corollary 6.7 is the assumption $\gamma_i = \gamma_{i,1} * \gamma_{i,2}$ with certain assumptions on $\gamma_{i,1}$ and $\gamma_{i,2}$. For a given function γ (or γ_i), it can be cumbersome to verify that it can be factorized as the convolution product of two such functions.

Hence, we close this section by providing more readily verifiable criteria which ensure that such a factorization is possible. For reasons that will become clear later, we begin with the following technical result:

Lemma 6.8. *For $\xi \in \mathbb{R}^d$, set $\{\xi\} := 1 + |\xi|^2$. Then, for each $\theta \in \mathbb{R}$ and each $\alpha \in \mathbb{N}_0^d$, there is a polynomial $P_{\theta,\alpha} \in \mathbb{R}[\xi_1, \dots, \xi_d]$ of degree $\deg P_{\theta,\alpha} \leq |\alpha|$ satisfying*

$$\partial^\alpha \{\xi\}^\theta = \{\xi\}^{\theta-|\alpha|} \cdot P_{\theta,\alpha}(\xi) \quad \forall \xi \in \mathbb{R}^d,$$

as well as $|P_{\theta,\alpha}(\xi)| \leq C \cdot (1 + |\xi|)^{|\alpha|}$ for all $\xi \in \mathbb{R}^d$, where

$$C = |\alpha|! \cdot [2 \cdot (1 + d + |\theta|)]^{|\alpha|}.$$

In particular, we have

$$|\partial^\alpha \{\xi\}^\theta| \leq 2^{|\theta|+|\alpha|} C \cdot (1 + |\xi|)^{2\theta-|\alpha|} \leq 2^{|\theta|+|\alpha|} C \cdot (1 + |\xi|)^{2\theta} \quad \forall \xi \in \mathbb{R}^d. \quad \blacktriangleleft$$

Proof. We prove existence of the polynomial $P_{\theta,\alpha}$ by induction on $N = |\alpha| \in \mathbb{N}_0$. But to do this, we need a slightly different formulation of the claim: For $P = \sum_{\sigma \in \mathbb{N}_0^d} c_\sigma \xi^\sigma \in \mathbb{R}[\xi_1, \dots, \xi_d]$, we define $\|P\|_* := \sum_{\sigma \in \mathbb{N}_0^d} |c_\sigma|$. Below, we will prove by induction on $N = |\alpha| \in \mathbb{N}_0$ that the polynomial $P_{\theta,\alpha}$ satisfying $\partial^\alpha \{\xi\}^\theta = \{\xi\}^{\theta-|\alpha|} \cdot P_{\theta,\alpha}(\xi)$ can

be chosen to satisfy $\|P_{\theta,\alpha}\|_* \leq C$ with C as in the statement of the lemma. This will imply the claim, since, for suitable coefficients $c_\sigma = c_\sigma(P_{\theta,\alpha})$, we have

$$\begin{aligned} |P_{\theta,\alpha}(\xi)| &\leq \sum_{|\sigma| \leq |\alpha|} |c_\sigma| \cdot |\xi^\sigma| \leq \sum_{|\sigma| \leq |\alpha|} |c_\sigma| \cdot (1 + |\xi|)^{|\sigma|} \\ &\leq (1 + |\xi|)^{|\alpha|} \cdot \|P_{\theta,\alpha}\|_* \leq C \cdot (1 + |\xi|)^{|\alpha|} \end{aligned}$$

for all $\xi \in \mathbb{R}^d$.

It remains to prove the modified claim by induction on N . For $N = 0$, all properties are trivially satisfied for $P_{\theta,\alpha} \equiv 1$.

For the induction step, we observe that each $\alpha \in \mathbb{N}_0^d$ with $|\alpha| = N + 1$ can be written as $\alpha = \beta + e_j$ for some $j \in \underline{d}$, where e_j is the j -th standard basis vector and where $\beta \in \mathbb{N}_0^d$ with $|\beta| = N$. Now, a direct calculation yields

$$\begin{aligned} \partial^\alpha \{\xi\}^\theta &= \partial_j \partial^\beta \{\xi\}^\theta = \partial_j \left[\{\xi\}^{\theta-|\beta|} \cdot P_{\theta,\beta}(\xi) \right] \\ &= \{\xi\}^{\theta-|\beta|} \cdot \partial_j P_{\theta,\beta}(\xi) + P_{\theta,\beta}(\xi) \cdot (\theta - |\beta|) \cdot \{\xi\}^{\theta-|\beta|-1} \cdot \partial_j \{\xi\} \\ &= \{\xi\}^{\theta-|\alpha|} [\{\xi\} \cdot \partial_j P_{\theta,\beta}(\xi) + 2(\theta - |\beta|) \cdot \xi_j \cdot P_{\theta,\beta}(\xi)] \\ &=: \{\xi\}^{\theta-|\alpha|} \cdot P_{\theta,\alpha}(\xi). \end{aligned}$$

Since $\deg[\{\xi\} \cdot \partial_j P_{\theta,\beta}] \leq 2 + \deg P_{\theta,\beta} - 1 \leq |\beta| + 1 = |\alpha|$, it is not hard to see that $\deg P_{\theta,\alpha} \leq |\alpha|$.

Next, observe for $\sigma \in \mathbb{N}_0^d$ with $\sigma_j \geq 1$ that

$$\partial_j \xi^\sigma = \prod_{\ell \neq j} \xi_\ell^{\sigma_\ell} \cdot \partial_j \xi_j^{\sigma_j} = \sigma_j \cdot \xi^{\sigma - e_j} \quad \forall \xi \in \mathbb{R}^d.$$

Furthermore, $\partial_j \xi^\sigma \equiv 0$ in case of $\sigma_j = 0$. For $P_{\theta,\beta}(\xi) = \sum_{|\sigma| \leq |\beta|} c_\sigma \xi^\sigma$, this implies

$$\|\partial_j P_{\theta,\beta}\|_* \leq \sum_{|\sigma| \leq |\beta|} |c_\sigma| \cdot \|\partial_j \xi^\sigma\|_* \leq \sum_{\substack{|\sigma| \leq |\beta| \\ \sigma_j \geq 1}} \sigma_j \cdot |c_\sigma| \leq |\beta| \cdot \|P_{\theta,\beta}\|_*.$$

Furthermore, since $\|\xi^\sigma \cdot P\|_* = \|P\|_*$ for each polynomial $P \in \mathbb{R}[\xi_1, \dots, \xi_d]$ and each $\sigma \in \mathbb{N}_0^d$ and since we have $\{\xi\} = 1 + \sum_{\ell=1}^d \xi^{2e_\ell}$, we get

$$\begin{aligned} \|P_{\theta,\alpha}\|_* &\leq \|\partial_j P_{\theta,\beta}\|_* + \sum_{\ell=1}^d \|\xi^{2e_\ell} \cdot \partial_j P_{\theta,\beta}\|_* + 2|\theta - |\beta|| \cdot \|\xi_j \cdot P_{\theta,\beta}\|_* \\ &\leq (1 + d) \cdot \|\partial_j P_{\theta,\beta}\|_* + 2(|\theta| + |\beta|) \cdot \|P_{\theta,\beta}\|_* \\ &\leq \|P_{\theta,\beta}\|_* [(1 + d) \cdot |\beta| + 2(|\theta| + |\beta|)] \\ &\leq \|P_{\theta,\beta}\|_* [(1 + d) \cdot (1 + |\beta|) + 2(1 + |\theta|)(1 + |\beta|)] \\ &\leq |\alpha| \cdot [(1 + d) + 2(1 + |\theta|)] \cdot \|P_{\theta,\beta}\|_* \\ &(\text{since } d \geq 1) \leq |\alpha| \cdot 2(1 + d + |\theta|) \cdot \|P_{\theta,\beta}\|_*. \end{aligned}$$

Since $\|P_{\theta,\beta}\|_* \leq |\beta|! \cdot [2(1 + d + |\theta|)]^{|\beta|}$ by induction and since $|\alpha| = |\beta| + 1$, the induction is complete.

It remains to verify the final statement of the lemma. To this end, note

$$\frac{1}{2} (1 + |\xi|)^2 \leq \{\xi\} = 1 + |\xi|^2 \leq (1 + |\xi|)^2 \leq 2 \cdot (1 + |\xi|)^2,$$

so that $\{\xi\}^\varrho \leq 2^{|\varrho|} \cdot (1 + |\xi|)^{2\varrho}$ for all $\varrho \in \mathbb{R}$ and $\xi \in \mathbb{R}^d$. Hence,

$$\begin{aligned} \left| \partial^\alpha \{\xi\}^\theta \right| &= \left| \{\xi\}^{\theta-|\alpha|} \cdot P_{\theta,\alpha}(\xi) \right| \leq 2^{|\theta-|\alpha||} \cdot (1 + |\xi|)^{2\theta-2|\alpha|} \cdot |P_{\theta,\alpha}(\xi)| \\ &\leq 2^{|\theta|+|\alpha|} C \cdot (1 + |\xi|)^{2\theta-|\alpha|}, \end{aligned}$$

as claimed. \square

Lemma 6.9. *Let $\varrho \in L^1(\mathbb{R}^d)$ with $\varrho \geq 0$. Let $N \geq d + 1$ and assume that $\gamma \in L^1(\mathbb{R}^d)$ satisfies $\hat{\gamma} \in C^N(\mathbb{R}^d)$ with*

$$|\partial^\alpha \hat{\gamma}(\xi)| \leq \varrho(\xi) \cdot (1 + |\xi|)^{-(d+1+\varepsilon)} \quad \forall \xi \in \mathbb{R}^d \quad \forall \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \leq N$$

for some $\varepsilon \in (0, 1]$.

Then there are functions $\gamma_1 \in C_0(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ and $\gamma_2 \in C^1(\mathbb{R}^d) \cap W^{1,1}(\mathbb{R}^d)$ with $\gamma = \gamma_1 * \gamma_2$ and with the following additional properties:

- (1) We have $\|\gamma_2\|_K \leq s_d \cdot 2^{1+d+3K} \cdot K! \cdot (1+d)^{1+2K}$ and $\|\nabla \gamma_2\|_K \leq \frac{s_d}{\varepsilon} \cdot 2^{4+d+3K} \cdot (1+d)^{2(1+K)} \cdot (K+1)!$ for all $K \in \mathbb{N}_0$, where as usual $\|g\|_K := \sup_{x \in \mathbb{R}^d} (1+|x|)^K |g(x)|$.
- (2) We have $\widehat{\gamma}_2 \in C^\infty(\mathbb{R}^d)$ with all partial derivatives of $\widehat{\gamma}_2$ being polynomially bounded (even bounded).
- (3) If $\widehat{\gamma} \in C^\infty(\mathbb{R}^d)$ with all partial derivatives being polynomially bounded, the same also holds for $\widehat{\gamma}_1$.
- (4) We have $\|\gamma_1\|_N \leq (1+d)^{1+2N} \cdot 2^{1+d+4N} \cdot N! \cdot \|\varrho\|_{L^1}$ and $\|\gamma\|_N \leq (1+d)^{N+1} \cdot \|\varrho\|_{L^1}$.
- (5) We have $|\partial^\alpha \widehat{\gamma}_1(\xi)| \leq 2^{1+d+4N} \cdot N! \cdot (1+d)^N \cdot \varrho(\xi)$ for all $\xi \in \mathbb{R}^d$ and $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq N$. ◀

Remark. For concrete special cases of \mathcal{Q} , this lemma will be applied as follows: In most cases, one can find a suitable function ϱ as above such that property (9) of Corollary 6.7 is satisfied as soon as all $\gamma_{j,1} \in L^1(\mathbb{R}^d)$ satisfy $|\partial^\alpha \widehat{\gamma}_{j,1}(\xi)| \lesssim \varrho(\xi)$ uniformly in $|\alpha| \leq N$, $j \in I$ and $\xi \in \mathbb{R}^d$.

In this case, the preceding lemma shows that if we instead assume $|\partial^\alpha \widehat{\gamma}_j(\xi)| \leq \varrho(\xi) \cdot (1+|\xi|)^{-(d+1+\varepsilon)}$ for all α, j, ξ as above, then we can write $\gamma_j = \gamma_{j,1} * \gamma_{j,2}$ such that $|\partial^\alpha \widehat{\gamma}_{j,1}(\xi)| \lesssim \varrho(\xi)$ uniformly in α, j, ξ as above, so that the family $(\gamma_{j,1})_{j \in I}$ satisfies property (9). Furthermore, the family $(\gamma_{j,2})_{j \in I}$ satisfies all assumptions of Corollary 6.7.

By possibly enlarging N for the application of the lemma, it is then not hard to ensure that all prerequisites of Corollary 6.7 are satisfied. Hence, Lemma 6.9 essentially solves the *factorization problem* mentioned before Lemma 6.8. ◆

Proof. We will use the notation $\{\xi\} = 1 + |\xi|^2$ from Lemma 6.8, as well as $\langle \xi \rangle := \{\xi\}^{1/2}$. With this notation, define $g \in C^\infty(\mathbb{R}^d)$ by $g : \mathbb{R}^d \rightarrow (0, \infty), \xi \mapsto \{\xi\}^{-\frac{d+1+\varepsilon}{2}} = \langle \xi \rangle^{-(d+1+\varepsilon)}$. In view of equation (1.9) and since $\langle \xi \rangle \geq \frac{1}{2}(1 + |\xi|)$, it is not hard to see $g \in L^1(\mathbb{R}^d)$, so that $\gamma_2 := \mathcal{F}^{-1}g \in C_0(\mathbb{R}^d)$ is well-defined.

Next, let $K \in \mathbb{N}_0$ be arbitrary. For $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq K$, Lemma 6.8 (with $\theta = -\frac{d+1+\varepsilon}{2}$) shows

$$|\partial^\alpha g(\xi)| \leq 2^{1+d+K} \cdot K! \cdot [4 \cdot (1+d)]^K \cdot (1+|\xi|)^{-(d+1+\varepsilon)} =: C_{d,K} \cdot (1+|\xi|)^{-(d+1+\varepsilon)} \quad \forall \xi \in \mathbb{R}^d. \quad (6.3)$$

In particular, this implies $g \in W^{K,1}(\mathbb{R}^d)$. In view of Lemma 6.3, we thus get

$$\begin{aligned} (1+|x|)^K \cdot |\gamma_2(x)| &= (1+|x|)^K \cdot |\mathcal{F}^{-1}g(x)| \leq (1+d)^K \cdot \left(|(\mathcal{F}^{-1}g)(x)| + \sum_{m=1}^d |[\mathcal{F}^{-1}(\partial_m^K g)](x)| \right) \\ &\quad (\text{since } |\mathcal{F}^{-1}h| \leq \|h\|_{L^1}) \leq (1+d)^K \cdot C_{d,K} \cdot \left\| (1+|\bullet|)^{-(d+1)} \right\|_{L^1} \cdot (1+d) \\ &\quad (\text{eq. (1.9)}) \leq s_d \cdot 2^{1+d+3K} \cdot K! \cdot (1+d)^{1+2K} \end{aligned}$$

for all $x \in \mathbb{R}^d$, and thus $\|\gamma_2\|_K \leq s_d \cdot 2^{1+d+3K} \cdot K! \cdot (1+d)^{1+2K} < \infty$ for arbitrary $K \in \mathbb{N}_0$, as desired. In particular, $\gamma_2 \in L^1(\mathbb{R}^d)$, so that $\widehat{\gamma}_2 = \mathcal{F}\mathcal{F}^{-1}g = g$ by Fourier inversion. Hence, equation (6.3) shows that all partial derivatives of $\widehat{\gamma}_2 = g$ are bounded.

Next, we want to estimate $\|\nabla \gamma_2\|_K$. To this end, we observe for arbitrary $j \in \underline{d}$ that

$$|\xi_j \cdot g(\xi)| \leq \{\xi\}^{1/2} \cdot |g(\xi)| = \{\xi\}^{-\frac{d+\varepsilon}{2}} = \langle \xi \rangle^{-(d+\varepsilon)} \in L^1(\mathbb{R}^d),$$

so that we can differentiate under the integral in the definition of $\gamma_2(x) = (\mathcal{F}^{-1}g)(x)$ to conclude for $g_j(\xi) := \xi_j \cdot g(\xi)$ that $\gamma_2 \in C^1(\mathbb{R}^d)$ with derivative

$$\partial_j \gamma_2(x) = 2\pi i \cdot \int_{\mathbb{R}^d} \xi_j \cdot g(\xi) \cdot e^{2\pi i \langle x, \xi \rangle} d\xi = 2\pi i \cdot (\mathcal{F}^{-1}g_j)(x).$$

Now, we want to apply Lemma 6.3 again to derive a bound for $\partial_j \gamma_2(x)$, which requires us to bound the derivatives $\partial^\alpha g_j$. To this end, we observe $\partial^\alpha \xi_j \equiv 0$ unless $\alpha = 0$, in which case we have $\partial^\alpha \xi_j = \xi_j$, or unless $\alpha = e_j$, in which

case we have $\partial^\alpha \xi_j = 1$. In combination with Leibniz's rule, this yields for $|\alpha| \leq K$ the estimate

$$\begin{aligned} |\partial^\alpha g_j(\xi)| &\leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} |\partial^\beta \xi_j| |\partial^{\alpha-\beta} g(\xi)| \\ &= \begin{cases} |\xi_j| \cdot |\partial^\alpha g(\xi)|, & \text{if } \alpha_j = 0, \\ |\xi_j| \cdot |\partial^\alpha g(\xi)| + \binom{\alpha}{e_j} \cdot |\partial^{\alpha-e_j} g(\xi)|, & \text{if } \alpha_j \geq 1 \end{cases} \\ \text{(eq. (6.3))} &\leq \begin{cases} C_{d,K} \cdot (1 + |\xi|)^{-(d+1+\varepsilon)} \cdot |\xi_j|, & \text{if } \alpha_j = 0, \\ C_{d,K} \cdot \left(|\xi_j| + \binom{\alpha}{e_j}\right) \cdot (1 + |\xi|)^{-(d+1+\varepsilon)}, & \text{if } \alpha_j \geq 1 \end{cases} \\ &\left(\text{since } \binom{\alpha}{e_j} = \binom{\alpha_j}{1} = \alpha_j \leq |\alpha| \leq K\right) \leq C_{d,K} \cdot (1 + K) \cdot (1 + |\xi|)^{-(d+\varepsilon)}. \end{aligned}$$

In particular, this implies $g_j \in W^{K,1}(\mathbb{R}^d)$. Hence, another application of Lemma 6.3 and equation (1.9) yields

$$\begin{aligned} (1 + |x|)^K \cdot |\partial_j \gamma_2(x)| &= 2\pi \cdot (1 + |x|)^K \cdot |\mathcal{F}^{-1} g_j(x)| \\ \text{(eq. (6.2))} &\leq 8 \cdot (1 + d)^K \cdot \left(|(\mathcal{F}^{-1} g_j)(x)| + \sum_{m=1}^d |[\mathcal{F}^{-1}(\partial_m^K g_j)](x)| \right) \\ &\leq 8 \cdot (1 + d)^K \cdot C_{d,K} \cdot (1 + K) \cdot (1 + d) \cdot \left\| (1 + |\bullet|)^{-(d+\varepsilon)} \right\|_{L^1} \\ \text{(eq. (1.9))} &\leq \frac{8d}{\varepsilon} \cdot 2^{4+d+3K} \cdot (1 + d)^{1+2K} \cdot (K + 1)! \end{aligned}$$

and hence $\|\nabla \gamma_2\|_K \leq \frac{8d}{\varepsilon} \cdot 2^{4+d+3K} \cdot (1 + d)^{2(1+K)} \cdot (K + 1)!$ for arbitrary $K \in \mathbb{N}_0$, as claimed. In particular, $\nabla \gamma_2 \in L^1(\mathbb{R}^d)$, so that $\gamma_2 \in C^1(\mathbb{R}^d) \cap W^{1,1}(\mathbb{R}^d)$, as claimed.

It remains to construct γ_1 with the desired properties. To this end, define $h : \mathbb{R}^d \rightarrow \mathbb{C}, \xi \mapsto \widehat{\gamma}(\xi) \cdot \{\xi\}^{\frac{d+1+\varepsilon}{2}}$, note $h \in C^N(\mathbb{R}^d)$ and observe that Lemma 6.8 shows for arbitrary $\beta \in \mathbb{N}_0^d$ with $|\beta| \leq N$ that

$$\left| \partial^\beta \{\xi\}^{\frac{d+1+\varepsilon}{2}} \right| \leq C_{d,N} \cdot (1 + |\xi|)^{d+1+\varepsilon} \quad (6.4)$$

with the same constant $C_{d,N}$ (with $N = K$) as in equation (6.3). In combination with Leibniz's rule and the d -dimensional binomial theorem (cf. [29, Section 8.1, Exercise 2.b]), this yields for arbitrary $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq N$ that

$$\begin{aligned} |\partial^\alpha h(\xi)| &\leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \cdot \left| \partial^\beta \{\xi\}^{\frac{d+1+\varepsilon}{2}} \right| \cdot |\partial^{\alpha-\beta} \widehat{\gamma}(\xi)| \\ \text{(assump. on } \widehat{\gamma} \text{ and eq. (6.4))} &\leq \varrho(\xi) \cdot (1 + |\xi|)^{-(d+1+\varepsilon)} C_{d,N} (1 + |\xi|)^{d+1+\varepsilon} \cdot \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \\ &\leq 2^N C_{d,N} \cdot \varrho(\xi) \in L^1(\mathbb{R}^d). \end{aligned} \quad (6.5)$$

In particular, $h \in L^1(\mathbb{R}^d)$, so that $\gamma_1 := \mathcal{F}^{-1} h \in C_0(\mathbb{R}^d)$ is well-defined. Furthermore, we get $h \in W^{N,1}(\mathbb{R}^d)$.

Hence, we can invoke Lemma 6.3 once again to derive

$$\begin{aligned} (1 + |x|)^N \cdot |\gamma_1(x)| &= (1 + |x|)^N \cdot |(\mathcal{F}^{-1} h)(x)| \\ \text{(eq. (6.2))} &\leq (1 + d)^N \cdot \left(|(\mathcal{F}^{-1} h)(x)| + \sum_{m=1}^d |[\mathcal{F}^{-1}(\partial_m^N h)](x)| \right) \\ &\leq (1 + d)^{N+1} \cdot 2^N C_{d,N} \cdot \|\varrho\|_{L^1} \\ &\leq (1 + d)^{1+2N} \cdot 2^{1+d+4N} \cdot N! \cdot \|\varrho\|_{L^1} < \infty \end{aligned}$$

for all $x \in \mathbb{R}^d$, so that $\|\gamma_1\|_N \leq (1 + d)^{1+2N} \cdot 2^{1+d+4N} \cdot N! \cdot \|\varrho\|_{L^1}$, as claimed. Since $N \geq d + 1$ by assumption, equation (1.9) implies in particular that $\gamma_1 \in L^1(\mathbb{R}^d)$. Hence, Fourier inversion yields $\widehat{\gamma}_1 = \mathcal{F} \mathcal{F}^{-1} h = h$, so that equation (6.5) yields the claimed estimate for $|\partial^\alpha \widehat{\gamma}_1|$.

Next, the convolution theorem yields

$$\mathcal{F}[\gamma_1 * \gamma_2](\xi) = \widehat{\gamma}_1(\xi) \cdot \widehat{\gamma}_2(\xi) = h(\xi) \cdot g(\xi) = \widehat{\gamma}(\xi) \quad \forall \xi \in \mathbb{R}^d$$

and thus $\gamma = \gamma_1 * \gamma_2$, by injectivity of the Fourier transform.

Next, note that if $\widehat{\gamma} \in C^\infty(\mathbb{R}^d)$ with all derivatives being polynomially bounded, we clearly get $\widehat{\gamma}_1 = h \in C^\infty(\mathbb{R}^d)$, again with all derivatives being polynomially bounded, thanks to Lemma 6.8 and the Leibniz rule.

It remains to establish the estimate for $\|\gamma\|_N$. But since $|\widehat{\gamma}| \leq \varrho \in L^1(\mathbb{R}^d)$, we get $\gamma = \mathcal{F}^{-1}\widehat{\gamma}$ by Fourier inversion. Furthermore, our assumptions easily yield $\partial^\alpha \widehat{\gamma} \in L^1(\mathbb{R}^d)$ for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq N$. Hence, a final application of Lemma 6.3, together with our assumptions on $\widehat{\gamma}$, yields

$$\begin{aligned} (1 + |x|)^N \cdot |\gamma(x)| &= (1 + |x|)^N \cdot |(\mathcal{F}^{-1}\widehat{\gamma})(x)| \\ &\leq (1 + d)^N \cdot \left(|(\mathcal{F}^{-1}\widehat{\gamma})(x)| + \sum_{m=1}^d |[\mathcal{F}^{-1}(\partial_m^N \widehat{\gamma})](x)| \right) \\ &\leq (1 + d)^N \cdot \left(\|\widehat{\gamma}\|_{L^1} + \sum_{m=1}^d \|\partial_m^N \widehat{\gamma}\|_{L^1} \right) \\ &\leq (1 + d)^{N+1} \cdot \|\varrho\|_{L^1} < \infty, \end{aligned}$$

which easily yields the claim. \square

7. EXISTENCE OF COMPACTLY SUPPORTED BANACH FRAMES AND ATOMIC DECOMPOSITIONS FOR α -MODULATION SPACES

In this section, we show that the general theory developed in this paper can be used to prove existence of compactly supported, structured Banach frames for α -modulation spaces. A brief discussion of the history and the applications of α -modulation spaces, as well as a comparison of our results with the established literature will be given at the end of the section.

We begin our considerations by recalling the definition of α -modulation spaces, as given by Borup and Nielsen[6]. First of all, we have to define the associated covering. Its admissibility was established in [6, Theorem 2.6]; precisely, the following was shown:

Theorem 7.1. (cf. [6, Theorem 2.6]) Let $d \in \mathbb{N}$ and $\alpha \in [0, 1)$ be arbitrary. Define $\alpha_0 := \frac{\alpha}{1-\alpha}$. Then there is a constant $r_1 = r_1(d, \alpha)$ such that the family

$$\mathcal{Q}^{(\alpha)} := \mathcal{Q}_r^{(\alpha)} := \left(Q_{r,k}^{(\alpha)} \right)_{k \in \mathbb{Z}^d \setminus \{0\}} := \left(B_{r \cdot |k|^{\alpha_0}}(|k|^{\alpha_0} k) \right)_{k \in \mathbb{Z}^d \setminus \{0\}} \quad (7.1)$$

is an admissible covering of \mathbb{R}^d for every $r > r_1$. The covering $\mathcal{Q}_r^{(\alpha)}$ is called the α -**modulation covering** of \mathbb{R}^d . If the values of r and α are clear from the context, we also write $Q_k := Q_{r,k}^{(\alpha)}$. \blacktriangleleft

The associated weight is defined in our next lemma. There, and in the remainder of this section, we use the notation $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$ for $\xi \in \mathbb{R}^d$.

Lemma 7.2. (cf. [77, Lemma 9.2]) Let $d \in \mathbb{N}$ and $\alpha \in [0, 1)$ and let $r > 0$ be chosen such that the α -modulation covering $\mathcal{Q}_r^{(\alpha)} = \left(Q_{r,k}^{(\alpha)} \right)_{k \in \mathbb{Z}^d \setminus \{0\}}$ is an admissible covering of \mathbb{R}^d . We then have

$$\langle \xi \rangle \asymp \langle k \rangle^{\frac{1}{1-\alpha}} \quad \text{for all } k \in \mathbb{Z}^d \setminus \{0\} \text{ and } \xi \in Q_{r,k}^{(\alpha)},$$

where the implied constant only depends on r, α .

Now, for $s \in \mathbb{R}$, we define the weight $w^{(s)}$ on $\mathbb{Z}^d \setminus \{0\}$ by

$$w_k^{(s)} := \langle k \rangle^s \quad \text{for } k \in \mathbb{Z}^d \setminus \{0\}.$$

Then $w^{(s)}$ is $\mathcal{Q}_r^{(\alpha)}$ -moderate (cf. equation (1.13)). \blacktriangleleft

Note that Theorem 7.1 only claims that $\mathcal{Q}_r^{(\alpha)}$ is an *admissible* covering of \mathbb{R}^d . The next result shows that it is actually a *structured* admissible covering of \mathbb{R}^d (cf. the remark after Assumption 6.1) and thus in particular a semi-structured covering.

Lemma 7.3. Let $d \in \mathbb{N}$ and $\alpha \in [0, 1)$ and let $r_1 = r_1(d, \alpha)$ be as in Theorem 7.1 and let $r > r_1$. For $k \in \mathbb{Z}^d \setminus \{0\}$, set $T_k := |k|^{\alpha_0} \cdot \text{id}$ and $b_k := |k|^{\alpha_0} k$ and let $Q := B_r(0)$. Then we have

$$\mathcal{Q}_r^{(\alpha)} = (T_k Q + b_k)_{k \in \mathbb{Z}^d \setminus \{0\}}$$

and with these choices, $\mathcal{Q}_r^{(\alpha)}$ is a semi-structured admissible covering of \mathbb{R}^d .

Finally, $\mathcal{Q}_r^{(\alpha)}$ admits a regular partition of unity $(\varphi_k)_{k \in \mathbb{Z}^d \setminus \{0\}}$ (which thus fulfills Assumption 6.1) and $\mathcal{Q}_r^{(\alpha)}$ fulfills the standing assumptions from Section 1.3; in particular, $\|T_k^{-1}\| \leq 1 =: \Omega_0$ for all $k \in \mathbb{Z}^d \setminus \{0\}$. ◀

Proof. The fact that $\mathcal{Q}_r^{(\alpha)}$ is a structured admissible covering of \mathbb{R}^d for $r > r_1$ was shown in [77, Lemma 9.3]. Since this is the case, [78, Theorem 2.8] shows that there is a regular partition of unity $\Phi = (\varphi_k)_{k \in \mathbb{Z}^d \setminus \{0\}}$ for $\mathcal{Q}_r^{(\alpha)}$. In view of Corollary 6.5, Φ is thus also a $\mathcal{Q}_r^{(\alpha)}$ - v_0 -BAPU for every weight v_0 satisfying the general assumptions of Section 1.3.

Finally, we clearly have $\|T_k^{-1}\| = \left\| |k|^{-\alpha_0} \text{id} \right\| = |k|^{-\alpha_0} \leq 1$ for all $k \in \mathbb{Z}^d \setminus \{0\}$, since $|k| \geq 1$ and $\alpha_0 = \frac{\alpha}{1-\alpha} \geq 0$. ◻

Finally, we introduce the weights $v = v^{(\mu)}$ that we will use for the weighted L^p spaces $L_v^p(\mathbb{R}^d)$.

Lemma 7.4. *For $\mu \in \mathbb{R}$ let*

$$\begin{aligned} v^{(\mu)} : \mathbb{R}^d &\rightarrow (0, \infty), x \mapsto \langle x \rangle^\mu = \left(1 + |x|^2\right)^{\mu/2}, \\ v_0 : \mathbb{R}^d &\rightarrow (0, \infty), x \mapsto [2 \cdot (1 + |x|)]^{|\mu|} \end{aligned}$$

and set $K := |\mu|$ and $\Omega_1 := 2^{|\mu|}$. With these choices, $v = v^{(\mu)}$ satisfies the standing assumptions of Section 1.3. ◀

Proof. First of all, assume $\mu = 1$. In this case, we get

$$\begin{aligned} v^{(\mu)}(x + y) &= \left| \left(\frac{1}{x + y} \right) \right| \leq 1 + |x + y| \\ &\leq 1 + |x| + |y| \leq (1 + |x|)(1 + |y|) \\ &\leq 2 \cdot \left| \left(\frac{1}{x} \right) \right| \cdot (1 + |y|) = v^{(\mu)}(x) \cdot v_0(y), \end{aligned}$$

where the last step used $\mu = 1$. Now, for arbitrary $\mu \geq 0$, we likewise get $v^{(\mu)}(x + y) \leq v^{(\mu)}(x) \cdot v_0(y)$ by taking the μ -th power of the preceding estimate.

Finally, if $\mu < 0$, we have

$$v^{(-\mu)}(x) = v^{(-\mu)}(x + y + (-y)) \leq v^{(-\mu)}(x + y) \cdot [2 \cdot (1 + |-y|)]^{|\mu|}.$$

Rearranging yields

$$v^{(\mu)}(x + y) = \left[v^{(-\mu)}(x + y) \right]^{-1} \leq \left[v^{(-\mu)}(x) \right]^{-1} \cdot [2 \cdot (1 + |y|)]^{|\mu|} = v^{(\mu)}(x) \cdot v_0(y).$$

Hence, we have shown for all $\mu \in \mathbb{R}$ that $v^{(\mu)}$ is v_0 -moderate.

It is clear that $v_0 \geq 1$ and that v_0 is symmetric. Furthermore, $v_0(x) = 2^{|\mu|} \cdot (1 + |x|)^{|\mu|} = \Omega_1 \cdot (1 + |x|)^K$ for all $x \in \mathbb{R}^d$. We do not necessarily have $K = 0$, but in Lemma 7.3, we already saw $\|T_k^{-1}\| \leq 1 = \Omega_0$ for all $k \in \mathbb{Z}^d \setminus \{0\}$.

The only thing which remains to be verified is that v_0 is submultiplicative. But we have

$$2 \cdot (1 + |x + y|) \leq 2 \cdot (1 + |x| + |y|) \leq 2 \cdot (1 + |x|)(1 + |y|) \leq 2 \cdot (1 + |x|) \cdot 2 \cdot (1 + |y|).$$

Taking the $|\mu|$ -th power of this estimate yields $v_0(x + y) \leq v_0(x) \cdot v_0(y)$, as desired. ◻

Having verified all these assumptions, we conclude from Proposition 2.24 and Lemma 5.5 that the α -modulation spaces defined below are indeed well-defined Quasi-Banach spaces.

Definition 7.5. For $d \in \mathbb{N}$ and $\alpha \in [0, 1)$, choose some $r > r_1(d, \alpha)$ with $r_1(d, \alpha)$ as in Theorem 7.1. Then, for $p, q \in (0, \infty]$ and $s, \mu \in \mathbb{R}$, we define the associated (**weighted**) α -modulation space as

$$M_{(s, \mu), \alpha}^{p, q}(\mathbb{R}^d) := \mathcal{D}\left(\mathcal{Q}_r^{(\alpha)}, L_{v^{(\mu)}}^p, \ell_{w^{(s/(1-\alpha))}}^q\right)$$

with $w^{(s/(1-\alpha))}$ and $v^{(\mu)}$ as in Lemmas 7.2 and 7.4, respectively.

Furthermore, we define the **classical** α -modulation space as $M_{s, \alpha}^{p, q}(\mathbb{R}^d) := M_{(s, 0), \alpha}^{p, q}(\mathbb{R}^d)$. ◀

Remark. • The classical α -modulation spaces $M_{s, \alpha}^{p, q}(\mathbb{R}^d)$ defined above coincide with the α -modulation spaces defined in [6, Definition 2.4], up to trivial identifications: The quasi-norms used in the two definitions are precisely the same; the only difference between the two definitions is that in [6, Definition 2.4], the α -modulation spaces are defined as subspaces of $\mathcal{S}'(\mathbb{R}^d)$. In contrast, with our definition as a decomposition space, $M_{s, \alpha}^{p, q}(\mathbb{R}^d)$ is a subspace of $Z'(\mathbb{R}^d) = [\mathcal{F}(C_c^\infty(\mathbb{R}^d))]'$, cf. Section 1.3. But [77, Lemma 9.15] and [77, Theorem 9.13] show

that each $f \in M_{s,\alpha}^{p,q}(\mathbb{R}^d)$ extends to a (uniquely determined) tempered distribution, which implies that the two different definitions of α -modulation spaces indeed yield the same spaces, up to trivial identifications.

- Observe that the parameter $r > r_1(d, \alpha)$ is suppressed on the left-hand side of the definition above. This is justified, as we show now: Since any two coverings $\mathcal{Q}_r^{(\alpha)}, \mathcal{Q}_t^{(\alpha)}$ (with $r, t > r_1(d, \alpha)$) use the same families $(T_k)_{k \in \mathbb{Z}^d \setminus \{0\}}$ and $(b_k)_{k \in \mathbb{Z}^d \setminus \{0\}}$, it follows that every regular partition of unity $\Phi = (\varphi_k)_{k \in \mathbb{Z}^d \setminus \{0\}}$ (cf. Assumption 6.1) for $\mathcal{Q}_r^{(\alpha)}$ is also a regular partition of unity for $\mathcal{Q}_t^{(\alpha)}$, at least for $t \geq r$, which we can always assume by symmetry. Thus, by choosing the *same* BAPU Φ for both coverings, we see $\mathcal{D}\left(\mathcal{Q}_r^{(\alpha)}, L_{v(\mu)}^p, \ell_{w(s^*)}^q\right) = \mathcal{D}\left(\mathcal{Q}_t^{(\alpha)}, L_{v(\mu)}^p, \ell_{w(s^*)}^q\right)$, with equivalent quasi-norms. Here, $s^* := s/(1-\alpha)$.

We finally note that this argument implicitly uses that different choices of the BAPU yield the same space (with equivalent quasi-norms), cf. Proposition 2.24. \blacklozenge

In the remainder of this section, we will determine conditions on the prototype γ which ensure that Corollary 6.6 (leading to Banach frames) or Corollary 6.7 (leading to atomic decompositions) is applicable to γ . We will see that this is the case for arbitrary Schwartz functions γ , as long as $\widehat{\gamma}$ fulfills a certain non-vanishing condition. To be precise, recall that in Corollaries 6.6 and 6.7, we allowed the prototype to depend on $i \in I$, i.e., we used a family $(\gamma_i)_{i \in I}$ of prototypes. But in this section, we will only consider the case where $\gamma_i = \gamma$ is independent of $i \in I$.

To begin with, we recall that in Corollary 6.6, we are imposing certain summability conditions on

$$M_{j,i} := \left(\frac{w_j^{(s/(1-\alpha))}}{w_i^{(s/(1-\alpha))}} \right)^\tau \cdot (1 + \|T_j^{-1}T_i\|)^\sigma \cdot \max_{|\beta| \leq 1} \left(|\det T_i|^{-1} \cdot \int_{Q_i} \max_{|\alpha| \leq N} \left| (\partial^\alpha \widehat{\partial^\beta \gamma})(S_j^{-1}\xi) \right| d\xi \right)^\tau$$

for suitable values of $\tau, \sigma > 0$ and $N \in \mathbb{N}$. To slightly simplify this expression, we will use the following notation for the remainder of the section:

$$s^* := s/(1-\alpha). \quad (7.2)$$

For our application of Corollary 6.6, we will assume $\gamma \in L^1(\mathbb{R}^d) \cap C^1(\mathbb{R}^d)$ with $\partial_\ell \gamma \in L^1(\mathbb{R}^d)$ for all $\ell \in \underline{d}$ and with $\widehat{\gamma} \in C^\infty(\mathbb{R}^d)$. Under these assumptions, elementary properties of the Fourier transform yield for $\beta = e_\ell$ (the ℓ -th unit vector) that

$$\widehat{\partial^\beta \gamma}(\xi) = 2\pi i \xi_\ell \cdot \widehat{\gamma}(\xi) \quad \forall \xi \in \mathbb{R}^d.$$

Since we clearly have $\left| \frac{\partial^\theta}{\partial \eta^\theta} \eta_\ell \right| \leq 1 + |\eta|$ for all $\eta \in \mathbb{R}^d$ and $\theta \in \mathbb{N}_0^d$, the Leibniz rule and the d -dimensional binomial theorem (cf. [29, Section 8.1, Exercise 2.b]) yield

$$\begin{aligned} \left| (\partial^\alpha \widehat{\partial^\beta \gamma})(\eta) \right| &= \left| 2\pi i \cdot \sum_{\theta \leq \alpha} \binom{\alpha}{\theta} \cdot \partial^\theta \eta_\ell \cdot (\partial^{\alpha-\theta} \widehat{\gamma})(\eta) \right| \\ &\leq 2\pi \cdot (1 + |\eta|) \cdot \sum_{\theta \leq \alpha} \binom{\alpha}{\theta} \cdot |(\partial^{\alpha-\theta} \widehat{\gamma})(\eta)| \\ &\stackrel{(\text{eq. (7.4)})}{\leq} (1 + |\eta|)^{1-N_0} \cdot 2\pi C \cdot \sum_{\theta \leq \alpha} \binom{\alpha}{\theta} \\ &\leq 2^{N+1} \pi \cdot C \cdot (1 + |\eta|)^{1-N_0}, \end{aligned} \quad (7.3)$$

where we used $|\alpha - \theta| \leq |\alpha| \leq N$ and assumed that there is some $N_0 \in \mathbb{N}$ satisfying

$$\max_{|\alpha| \leq N} |(\partial^\alpha \widehat{\gamma})(\eta)| \leq C \cdot (1 + |\eta|)^{-N_0} \quad \forall \eta \in \mathbb{R}^d. \quad (7.4)$$

Recall that equation (7.3) holds for $\beta = e_\ell$, with arbitrary $\ell \in \underline{d}$. But for $\beta = 0$, we simply have

$$\left| (\partial^\alpha \widehat{\partial^\beta \gamma})(\eta) \right| = |(\partial^\alpha \widehat{\gamma})(\eta)| \leq C \cdot (1 + |\eta|)^{-N_0} \leq 2^{N+1} \pi \cdot C \cdot (1 + |\eta|)^{1-N_0},$$

so that we have verified equation (7.3) for arbitrary $\alpha, \beta \in \mathbb{N}_0^d$ with $|\beta| \leq 1$ and $|\alpha| \leq N$.

Hence, we have shown for $C' := 2^{N+1} \pi \cdot C$ that

$$M_{j,i} \leq (C')^\tau \cdot \left(\frac{w_j^{(s^*)}}{w_i^{(s^*)}} \right)^\tau \cdot (1 + \|T_j^{-1}T_i\|)^\sigma \cdot \left(|\det T_i|^{-1} \cdot \int_{Q_i} (1 + |S_j^{-1}\xi|)^{1-N_0} d\xi \right)^\tau =: (C')^\tau \cdot M_{j,i}^{(0)} \quad (7.5)$$

for all $i, j \in \mathbb{Z}^d \setminus \{0\}$. In view of this estimate, the following lemma is crucial:

Lemma 7.6. *Let $d \in \mathbb{N}$ and $\alpha \in [0, 1)$ and set $\alpha_0 := \frac{\alpha}{1-\alpha}$. With $\mathcal{Q} = \mathcal{Q}_r^{(\alpha)}$ and with $M_{j,i}^{(0)}$ as in equation (7.5), assume*

$$N_0 \geq d + 2 + \frac{d+1}{\tau} + \max \left\{ |s^* + d\alpha_0|, \left| s^* + \left(d - \frac{\sigma}{\tau} \right) \alpha_0 \right| \right\}.$$

Then we have

$$\sup_{i \in \mathbb{Z}^d \setminus \{0\}} \sum_{j \in \mathbb{Z}^d \setminus \{0\}} M_{j,i}^{(0)} \leq \Omega \quad \text{and} \quad \sup_{j \in \mathbb{Z}^d \setminus \{0\}} \sum_{i \in \mathbb{Z}^d \setminus \{0\}} M_{j,i}^{(0)} \leq \Omega$$

for

$$\Omega := 6^d 2^{1+\sigma+\tau|s^*|} \cdot \max \left\{ 4^{\alpha_0(\sigma+d\tau)+\tau|s^*|} \cdot (12^{N_0} s_d)^\tau, (2+4r)^{\tau|s^*|+\alpha_0[\sigma+\tau(d+N_0)]} \cdot 2^{\tau d} \cdot (1+(2+4r)^{\alpha_0} \cdot r)^{\tau(N_0+d)} \right\}. \quad \blacktriangleleft$$

Proof. For brevity, set $M := N_0 - 1$. Recall $T_j = |j|^{\alpha_0} \cdot \text{id}$ and $b_j = |j|^{\alpha_0} j$, so that

$$S_j^{-1} \xi = T_j^{-1} (\xi - b_j) = |j|^{-\alpha_0} (\xi - |j|^{\alpha_0} j) = |j|^{-\alpha_0} \xi - j.$$

Hence,

$$\begin{aligned} \int_{Q_i} (1 + |S_j^{-1} \xi|)^{1-N_0} d\xi &= \int_{B_{|i|^{\alpha_0} r}(|i|^{\alpha_0} i)} (1 + |j|^{-\alpha_0} \xi - j)^{-M} d\xi \\ &\quad (\text{with } \eta = |j|^{-\alpha_0} \xi) = |j|^{d\alpha_0} \cdot \int_{B_{(|i|/|j|)^{\alpha_0} \cdot r}((|i|/|j|)^{\alpha_0} i)} (1 + |\eta - j|)^{-M} d\eta \\ &\quad (\text{with } \xi = \eta - j) = |j|^{d\alpha_0} \cdot \int_{B_{(|i|/|j|)^{\alpha_0} \cdot r}((|i|/|j|)^{\alpha_0} i - j)} (1 + |\xi|)^{-M} d\xi \\ &= |j|^{d\alpha_0} \cdot \int_{B_{R_{i,j}}(\xi_{i,j})} (1 + |\xi|)^{-M} d\xi, \end{aligned}$$

where we defined

$$\xi_{i,j} := \left(\frac{|i|}{|j|} \right)^{\alpha_0} i - j \quad \text{and} \quad R_{i,j} := \left(\frac{|i|}{|j|} \right)^{\alpha_0} \cdot r$$

for $i, j \in I = \mathbb{Z}^d \setminus \{0\}$. Here, $r > r_1(d, \alpha)$ comes from the covering $\mathcal{Q}_r^{(\alpha)}$.

All in all, since $\|T_j^{-1} T_i\| = \||i|^{\alpha_0} / |j|^{\alpha_0} \cdot \text{id}\| = (|i|/|j|)^{\alpha_0}$ and $|\det T_i| = |\det |i|^{\alpha_0} \text{id}| = |i|^{d\alpha_0}$, and because of $|i| \leq \langle i \rangle \leq 1 + |i| \leq 2|i|$ for $i \in \mathbb{Z}^d$, so that $\frac{1}{2} \frac{|i|}{|j|} \leq \frac{\langle j \rangle}{\langle i \rangle} \leq 2 \frac{|i|}{|j|}$, we get the following estimate for $M_{j,i}^{(0)}$:

$$\begin{aligned} M_{j,i}^{(0)} &= \left(\frac{w_j^{(s^*)}}{w_i^{(s^*)}} \right)^\tau \cdot (1 + \|T_j^{-1} T_i\|)^\sigma \cdot \left(|\det T_i|^{-1} \cdot \int_{Q_i} (1 + |S_j^{-1} \xi|)^{1-N_0} d\xi \right)^\tau \\ &= \left(\frac{\langle j \rangle}{\langle i \rangle} \right)^{s^* \cdot \tau} \cdot \left(1 + \left(\frac{|i|}{|j|} \right)^{\alpha_0} \right)^\sigma \cdot \left[\left(\frac{|j|}{|i|} \right)^{d\alpha_0} \cdot \int_{B_{R_{i,j}}(\xi_{i,j})} (1 + |\xi|)^{-M} d\xi \right]^\tau \\ &\quad (\text{since } (1+a)^\sigma \leq 2^{\sigma \cdot (1+a^\sigma)}) \leq 2^{\sigma+\tau|s^*|} \cdot \left(\frac{|j|}{|i|} \right)^{\tau(s^*+d\alpha_0)} \cdot \left(1 + \left(\frac{|j|}{|i|} \right)^{-\sigma\alpha_0} \right) \cdot \left(\int_{B_{R_{i,j}}(\xi_{i,j})} (1 + |\xi|)^{-M} d\xi \right)^\tau \\ &= 2^{\sigma+\tau|s^*|} \cdot \sum_{\lambda \in \{0,1\}} \left[\left(\frac{|j|}{|i|} \right)^{k_\lambda} \cdot \left(\int_{B_{R_{i,j}}(\xi_{i,j})} (1 + |\xi|)^{-M} d\xi \right)^\tau \right], \end{aligned} \quad (7.6)$$

where we defined $k_\lambda := \tau(s^* + d\alpha_0) - \lambda\sigma\alpha_0$ for $\lambda \in \{0, 1\}$.

Thus, our main goal is to estimate the term

$$\Xi_{i,j}^{(k)} := \left(\frac{|j|}{|i|} \right)^k \cdot \left(\int_{B_{R_{i,j}}(\xi_{i,j})} (1 + |\xi|)^{-M} d\xi \right)^\tau$$

for arbitrary $i, j \in \mathbb{Z}^d \setminus \{0\}$, $k \in \mathbb{R}$ and $\tau > 0$. To this end, we will distinguish three cases concerning i, j below. But before that, we introduce a useful notation and some related estimates that will be used in several of the cases: For $x \in \mathbb{R}^d$, we set $[x] := 1 + |x|$. We then have

$$[x]^z \leq [y]^z \cdot [x - y]^{|z|} \quad \forall z \in \mathbb{R} \quad \forall x, y \in \mathbb{R}^d. \quad (7.7)$$

Indeed, since $[x] = 1 + |x| \leq 1 + |y| + |x - y| \leq (1 + |y|)(1 + |x - y|) = [y] \cdot [x - y]$, we get the claim for $z \geq 0$. Finally, for $z < 0$, we have

$$[x]^z = [x]^{-|z|} = \left([x]^{-|z|} [x - y]^{-|z|}\right) \cdot [x - y]^{|z|}$$

$$\text{(eq. (7.7) rearranged, with } x, y \text{ interchanged and } |z| \text{ instead of } z) \leq [y]^{-|z|} \cdot [x - y]^{|z|} = [y]^z \cdot [x - y]^{|z|},$$

as desired. Now, note for $i, j \in \mathbb{Z}^d \setminus \{0\}$ that $|i| \leq [i] \leq 2|i|$ and likewise for j , so that

$$|i|^z \leq 2^{|z|} \cdot [i]^z \leq 2^{|z|} \cdot [j]^z \cdot [i - j]^{|z|} \leq 4^{|z|} \cdot |j|^z \cdot [i - j]^{|z|}. \quad (7.8)$$

We will also need the following estimate, which I learned from [52]:

$$|\beta \cdot x - y| \geq |x - y| \quad \text{if } \beta \in \mathbb{R}_{\geq 1} \text{ and } x, y \in \mathbb{R}^d \text{ with } |x| \geq |y|. \quad (7.9)$$

For $\beta = 1$, this estimate is trivial, so that we can assume $\beta > 1$. Next, note that both sides are nonnegative, so that the estimate is equivalent to $|\beta \cdot x - y|^2 \geq |x - y|^2$ and thus to

$$\begin{aligned} \beta^2 |x|^2 - 2\beta \cdot \langle x, y \rangle + |y|^2 &\stackrel{!}{\geq} |x|^2 - 2 \langle x, y \rangle + |y|^2 \\ \iff |x|^2 \cdot (\beta^2 - 1) &\stackrel{!}{\geq} 2 \cdot \langle x, y \rangle \cdot (\beta - 1) \\ \text{(since } \beta - 1 > 0) \iff |x|^2 \cdot (\beta + 1) &\stackrel{!}{\geq} 2 \cdot \langle x, y \rangle. \end{aligned}$$

But the Cauchy-Schwarz inequality yields $2 \cdot \langle x, y \rangle \leq 2 \cdot |\langle x, y \rangle| \leq 2 \cdot |x| |y| \leq 2 \cdot |x|^2 \leq (1 + \beta) \cdot |x|^2$, since $|y| \leq |x|$ and since $\beta \geq 1$. Hence, we have established equation (7.9). We remark that for this estimate, it is crucial to use a norm which is induced by a scalar product. For other norms, equation (7.9) can fail.

Now, we distinguish three cases depending on i, j :

Case 1: We have $|i| \geq 2|j| + 4r$. In this case, we get

$$\begin{aligned} \left| \left(\frac{|i|}{|j|} \right)^{\alpha_0} i \right| &\leq \left| \left(\frac{|i|}{|j|} \right)^{\alpha_0} i - j \right| + |j| \\ &\leq |\xi_{i,j}| + \frac{|j|}{2} \\ \text{(since } \frac{|i|}{|j|} \geq 2 \geq 1 \text{ and } \alpha_0 \geq 0) &\leq |\xi_{i,j}| + \frac{1}{2} \left| \left(\frac{|i|}{|j|} \right)^{\alpha_0} i \right| \end{aligned}$$

and thus, since $|i| \geq 4r$,

$$|\xi_{i,j}| \geq \frac{1}{2} \left| \left(\frac{|i|}{|j|} \right)^{\alpha_0} i \right| \geq 2r \cdot \left(\frac{|i|}{|j|} \right)^{\alpha_0} = 2 \cdot R_{i,j}. \quad (7.10)$$

Hence, for arbitrary $\xi \in B_{R_{i,j}}(\xi_{i,j})$, we have $|\xi| \geq |\xi_{i,j}| - |\xi - \xi_{i,j}| \geq |\xi_{i,j}| - R_{i,j} \geq \frac{1}{2} |\xi_{i,j}|$ and thus

$$\begin{aligned} \int_{B_{R_{i,j}}(\xi_{i,j})} (1 + |\xi|)^{-M} d\xi &\leq \left[\sup_{\xi \in B_{R_{i,j}}(\xi_{i,j})} (1 + |\xi|)^{d+1-M} \right] \cdot \int_{B_{R_{i,j}}(\xi_{i,j})} (1 + |\xi|)^{-(d+1)} d\xi \\ \text{(since } d+1-M \leq 0) &\leq \left(1 + \frac{1}{2} |\xi_{i,j}| \right)^{d+1-M} \cdot \int_{\mathbb{R}^d} (1 + |\xi|)^{-(d+1)} d\xi \\ \text{(eq. (1.9))} &\leq 2^M \cdot s_d \cdot |\xi_{i,j}|^{d+1-M} \\ \text{(eq. (7.10) and } d+1-M \leq 0) &\leq 4^M \cdot s_d \cdot \left| \left(\frac{|i|}{|j|} \right)^{\alpha_0} i \right|^{d+1-M} \\ \text{(since } d+1-M \leq 0 \text{ and } \frac{|i|}{|j|} \geq 1) &\leq 4^M \cdot s_d \cdot |i|^{d+1-M}. \end{aligned} \quad (7.11)$$

Next, we observe $|i| \geq 2|j| + 4r \geq |j|$ and $i \in \mathbb{Z}^d \setminus \{0\}$, so that $|i| \geq 1$. This implies

$$|i - j| = 1 + |i - j| \leq 1 + |i| + |j| \leq 1 + 2|i| \leq 3|i|,$$

so that we finally arrive, again using $d + 1 - M \leq 0$, at

$$\int_{B_{R_{i,j}}(\xi_{i,j})} (1 + |\xi|)^{-M} d\xi \leq 12^M \cdot s_d \cdot [i - j]^{d+1-M}.$$

Thus, using equation (7.8), we conclude

$$\Xi_{i,j}^{(k)} = \left(\frac{|j|}{|i|} \right)^k \cdot \left(\int_{B_{R_{i,j}}(\xi_{i,j})} (1 + |\xi|)^{-M} d\xi \right)^\tau \leq 4^{|k|} \cdot (12^M \cdot s_d)^\tau \cdot [j - i]^{|k| + \tau(d+1-M)}. \quad (7.12)$$

Case 2: We have $|j| \geq 2|i| + 4r$. Here, we first observe $|j - i| \geq |j| - |i| \geq |i| + 4r \geq 4r$. Hence, we get

$$\begin{aligned} |\xi_{i,j}| &= \left(\frac{|i|}{|j|} \right)^{\alpha_0} \cdot \left| \left(\frac{|j|}{|i|} \right)^{\alpha_0} j - i \right| \\ &\stackrel{(\text{eq. (7.9) and } (|j|/|i|)^{\alpha_0} \geq 1, \text{ as well as } |j| \geq |i|)}{\geq} \left(\frac{|i|}{|j|} \right)^{\alpha_0} \cdot |j - i| \\ &\geq 4 \cdot \left(\frac{|i|}{|j|} \right)^{\alpha_0} r = 4 \cdot R_{i,j}. \end{aligned}$$

Further, $|j| \geq 2|i| + 4r \geq 2|i|$ implies $|i| \leq \frac{|j|}{2}$ and thus $|j| - |i| \geq \frac{1}{2}|j|$. Consequently,

$$|j - i| \leq |j| + |i| \leq 2|j| \leq 4 \cdot (|j| - |i|),$$

so that we get

$$\begin{aligned} |\xi_{i,j}| &= \left| \left(\frac{|i|}{|j|} \right)^{\alpha_0} i - j \right| \\ &\geq |j| - \left(\frac{|i|}{|j|} \right)^{\alpha_0} |i| \\ &\stackrel{(\text{since } |i|/|j| \leq 1)}{\geq} |j| - |i| \geq \frac{|j - i|}{4}. \end{aligned}$$

Now, for arbitrary $\xi \in B_{R_{i,j}}(\xi_{i,j})$, the two preceding displayed estimates yield $|\xi| \geq |\xi_{i,j}| - R_{i,j} \geq \frac{3}{4}|\xi_{i,j}| \geq \frac{3}{16}|j - i|$ and hence $1 + |\xi| \geq \frac{3}{16} \cdot [j - i]$. With an estimate entirely analogous to that in equation (7.11), this implies

$$\begin{aligned} \int_{B_{R_{i,j}}(\xi_{i,j})} (1 + |\xi|)^{-M} d\xi &\leq \left[\sup_{\xi \in B_{R_{i,j}}(\xi_{i,j})} (1 + |\xi|)^{d+1-M} \right] \cdot \int_{B_{R_{i,j}}(\xi_{i,j})} (1 + |\xi|)^{-(d+1)} d\xi \\ &\stackrel{(\text{eq. (1.9) and } d+1-M \leq 0)}{\leq} s_d \cdot \left(\frac{16}{3} \right)^M \cdot [j - i]^{d+1-M}. \end{aligned}$$

In view of equation (7.8) and since $\frac{16}{3} \leq 12$, we conclude

$$\Xi_{i,j}^{(k)} = \left(\frac{|j|}{|i|} \right)^k \cdot \left(\int_{B_{R_{i,j}}(\xi_{i,j})} (1 + |\xi|)^{-M} d\xi \right)^\tau \leq 4^{|k|} \cdot (12^M \cdot s_d)^\tau \cdot [j - i]^{|k| + \tau(d+1-M)},$$

as in the previous case.

Case 3: The remaining case, i.e., $|i| < 2|j| + 4r$ and $|j| < 2|i| + 4r$. Since $i, j \in \mathbb{Z}^d \setminus \{0\}$, we have $|i|, |j| \geq 1$ and thus $|i| \leq |j| \cdot (2 + 4r)$ and $|j| \leq |i| \cdot (2 + 4r)$. In particular, we have

$$R_{i,j} = \left(\frac{|i|}{|j|} \right)^{\alpha_0} \cdot r \leq r \cdot (2 + 4r)^{\alpha_0} =: C_{r,\alpha_0},$$

so that every $\xi \in B_{R_{i,j}}(\xi_{i,j})$ satisfies

$$\begin{aligned} 1 + |\xi_{i,j}| &\leq 1 + |\xi_{i,j} - \xi| + |\xi| \leq 1 + R_{i,j} + |\xi| \\ &\leq (1 + R_{i,j})(1 + |\xi|) \leq (1 + C_{r,\alpha_0})(1 + |\xi|). \end{aligned}$$

Consequently,

$$\begin{aligned} \int_{B_{R_{i,j}}(\xi_{i,j})} (1 + |\xi|)^{-M} d\xi &\leq (1 + C_{r,\alpha_0})^M \cdot \lambda_d(B_{R_{i,j}}(\xi_{i,j})) \cdot (1 + |\xi_{i,j}|)^{-M} \\ &\stackrel{(\text{since } \lambda_d(B_1(0)) \leq \lambda_d([-1,1]^d) = 2^d)}{\leq} 2^d \cdot (1 + C_{r,\alpha_0})^M R_{i,j}^d \cdot (1 + |\xi_{i,j}|)^{-M} \\ &\stackrel{(\text{since } R_{i,j} \leq C_{r,\alpha_0})}{\leq} 2^d \cdot (1 + C_{r,\alpha_0})^{M+d} \cdot (1 + |\xi_{i,j}|)^{-M} =: C_{d,M,r,\alpha_0} \cdot (1 + |\xi_{i,j}|)^{-M}. \end{aligned}$$

Furthermore, since $\frac{1}{2+4r} \leq \frac{|j|}{|i|} \leq 2+4r$, we get $\left(\frac{|j|}{|i|}\right)^k \leq (2+4r)^{|k|}$ and thus

$$\Xi_{i,j}^{(k)} \leq (2+4r)^{|k|} \cdot C_{d,M,r,\alpha_0}^\tau \cdot (1+|\xi_{i,j}|)^{-\tau M}.$$

To further estimate the right-hand side, we distinguish two sub-cases:

(1) We have $|i| \geq |j|$. In this case, equation (7.9) yields

$$|\xi_{i,j}| = \left| \left(\frac{|i|}{|j|} \right)^{\alpha_0} i - j \right| \geq |i - j|$$

and hence

$$\begin{aligned} \Xi_{i,j}^{(k)} &\leq (2+4r)^{|k|} \cdot C_{d,M,r,\alpha_0}^\tau \cdot [i-j]^{-\tau M} \\ &\leq (2+4r)^{|k|} \cdot C_{d,M,r,\alpha_0}^\tau \cdot [i-j]^{|k|+\tau(d+1-M)}. \end{aligned}$$

(2) We have $|j| \geq |i|$. In this case, we can again—after some rearranging—use equation (7.9) to obtain

$$\begin{aligned} |\xi_{i,j}| &= \left| \left(\frac{|i|}{|j|} \right)^{\alpha_0} i - j \right| = \left(\frac{|i|}{|j|} \right)^{\alpha_0} \left| \left(\frac{|j|}{|i|} \right)^{\alpha_0} j - i \right| \\ (\text{eq. (7.9)}) &\geq (2+4r)^{-\alpha_0} \cdot |j - i| \end{aligned}$$

and hence

$$\begin{aligned} \Xi_{i,j}^{(k)} &\leq (2+4r)^{|k|} \cdot C_{d,M,r,\alpha_0}^\tau \cdot \left(1 + (2+4r)^{-\alpha_0} \cdot |j - i| \right)^{-\tau M} \\ &\leq (2+4r)^{|k|+\alpha_0\tau M} \cdot C_{d,M,r,\alpha_0}^\tau \cdot [j-i]^{-\tau M} \\ &\leq (2+4r)^{|k|+\alpha_0\tau M} \cdot C_{d,M,r,\alpha_0}^\tau \cdot [j-i]^{|k|+\tau(d+1-M)}. \end{aligned}$$

All in all, the preceding case distinction has established the bound

$$\begin{aligned} \Xi_{i,j}^{(k)} &\leq \max \left\{ 4^{|k|} \cdot (12^M \cdot s_d)^\tau, (2+4r)^{|k|+\alpha_0\tau M} \cdot C_{d,M,r,\alpha_0}^\tau \cdot [j-i]^{|k|+\tau(d+1-M)} \right\} \\ &=: C_{d,M,r,\alpha_0,k,\tau} \cdot [j-i]^{|k|+\tau(d+1-M)} \end{aligned} \quad (7.13)$$

for all $i, j \in \mathbb{Z}^d \setminus \{0\}$ and all $k \in \mathbb{R}$, with $C_{d,M,r,\alpha_0} = 2^d \cdot (1 + C_{r,\alpha_0})^{M+d}$ and $C_{r,\alpha_0} = r \cdot (2+4r)^{\alpha_0}$.

Now, we want to utilize this estimate for $k = k_\lambda = \tau(s^* + d\alpha_0) - \lambda\sigma\alpha_0$ for $\lambda \in \{0, 1\}$, cf. equation (7.6). Note the equivalence

$$\begin{aligned} |k_\lambda| + \tau(d+1-M) &= |k_\lambda| + \tau(d+2-N_0) \stackrel{!}{\leq} -(d+1) \\ \iff N_0 &\stackrel{!}{\geq} d+2 + \frac{|k_\lambda| + d+1}{\tau}, \end{aligned}$$

where the last condition is satisfied for $\lambda \in \{0, 1\}$ by definition of k_λ and our assumptions regarding N_0 . Hence, we get—in view of equations (5.10) and (7.13) and because of $|j-i| \geq \|j-i\|_\infty$ —that

$$\begin{aligned} \sum_{j \in \mathbb{Z}^d \setminus \{0\}} \Xi_{i,j}^{(k_\lambda)} &\leq \left[\max_{\lambda \in \{0,1\}} C_{d,M,r,\alpha_0,k_\lambda,\tau} \right] \cdot \sum_{j \in \mathbb{Z}^d} [j-i]^{-(d+1)} \\ (\text{with } \ell=j-i) &\leq \left[\max_{\lambda \in \{0,1\}} C_{d,M,r,\alpha_0,k_\lambda,\tau} \right] \cdot \sum_{\ell \in \mathbb{Z}^d} (1 + \|\ell\|_\infty)^{-(d+1)} \leq 6^d \cdot \max_{\lambda \in \{0,1\}} C_{d,M,r,\alpha_0,k_\lambda,\tau}. \end{aligned}$$

The same estimate also holds when taking the sum over $i \in \mathbb{Z}^d \setminus \{0\}$ instead of over $j \in \mathbb{Z}^d \setminus \{0\}$. In view of equation (7.6), we thus get

$$\sup_{i \in \mathbb{Z}^d \setminus \{0\}} \sum_{j \in \mathbb{Z}^d \setminus \{0\}} M_{j,i}^{(0)} \leq 2^{1+\sigma+\tau|s^*|} \cdot 6^d \cdot \max_{\lambda \in \{0,1\}} C_{d,M,r,\alpha_0,k_\lambda,\tau}$$

and the same estimate also holds for $\sup_{j \in \mathbb{Z}^d \setminus \{0\}} \sum_{i \in \mathbb{Z}^d \setminus \{0\}} M_{j,i}^{(0)}$. This easily yields the claim. \square

Now, we can derive readily verifiable conditions which ensure that the structured family generated by γ yields a Banach frame for a given α -modulation space.

Theorem 7.7. Let $d \in \mathbb{N}$, $\alpha \in [0, 1)$ and choose $r > r_1(d, \alpha)$ with $r_1(d, \alpha)$ as in Theorem 7.1.

Let $s_0, \mu_0 \geq 0$ and $p_0, q_0 \in (0, 1]$, as well as $\varepsilon \in (0, 1)$. Assume that $\gamma : \mathbb{R}^d \rightarrow \mathbb{C}$ satisfies the following:

- (1) We have $\gamma \in L^1_{(1+|\bullet|)^{\mu_0}}(\mathbb{R}^d)$ and $\widehat{\gamma} \in C^\infty(\mathbb{R}^d)$, where all partial derivatives of $\widehat{\gamma}$ are polynomially bounded.
- (2) We have $\gamma \in C^1(\mathbb{R}^d)$ and $\partial_\ell \gamma \in L^1_{(1+|\bullet|)^{\mu_0}}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ for all $\ell \in \underline{d}$.
- (3) We have $|\widehat{\gamma}(\xi)| \geq c > 0$ for all $\xi \in \overline{B_r}(0)$.
- (4) We have

$$|(\partial^\beta \widehat{\gamma})(\xi)| \lesssim (1 + |\xi|)^{-N_0}$$

for all $\xi \in \mathbb{R}^d$ and all $\beta \in \mathbb{N}_0^d$ with $|\beta| \leq \left\lceil \mu_0 + \frac{d+\varepsilon}{p_0} \right\rceil$, where

$$N_0 = d + 2 + \frac{d+1}{\min\{p_0, q_0\}} + \frac{1}{1-\alpha} \cdot \max \left\{ s_0 + \alpha d, s_0 + \alpha \left(\frac{d}{p_0} - d + \mu_0 + \left\lceil \mu_0 + \frac{d+\varepsilon}{p_0} \right\rceil \right) \right\}.$$

Then there is some $\delta_0 > 0$ such that for all $0 < \delta \leq \delta_0$, the family

$$\Gamma^{(\delta)} := \left(L_{\delta \cdot k/|i|^{\alpha_0}} \widehat{\gamma^{[i]}} \right)_{i \in \mathbb{Z}^d \setminus \{0\}, k \in \mathbb{Z}^d}, \quad \text{with} \quad \gamma^{[i]} = |i|^{\frac{d-\alpha_0}{2}} \cdot M_{|i|^{\alpha_0} \cdot i} [\gamma \circ |i|^{\alpha_0} \text{id}] \quad \text{and} \quad \tilde{g}(x) := g(-x)$$

forms a Banach frame for $M_{(s,\mu),\alpha}^{p,q}(\mathbb{R}^d)$ for all $|s| \leq s_0$, $|\mu| \leq \mu_0$ and all $p, q \in (0, \infty]$ with $p \geq p_0$ and $q \geq q_0$.

Precisely, this means the following: Define the coefficient space

$$\mathcal{C}_{p,q,s,\mu}^{(\alpha)} := \ell^q_{\left[|i|^{\frac{1}{1-\alpha}(s+\alpha d(\frac{1}{2}-\frac{1}{p}))}\right]}_{i \in \mathbb{Z}^d \setminus \{0\}} \left(\left[\ell^p_{[(1+|k|/|i|^{\alpha_0})^\mu]_{k \in \mathbb{Z}^d}}(\mathbb{Z}^d) \right]_{i \in \mathbb{Z}^d \setminus \{0\}} \right).$$

Then the following hold:

- (1) The **analysis map**

$$A^{(\delta)} : M_{(s,\mu),\alpha}^{p,q}(\mathbb{R}^d) \rightarrow \mathcal{C}_{p,q,s,\mu}^{(\alpha)}, f \mapsto \left[\left(\gamma^{[i]} * f \right) (\delta \cdot k / |i|^{\alpha_0}) \right]_{i \in \mathbb{Z}^d \setminus \{0\}, k \in \mathbb{Z}^d}$$

is well-defined and bounded for all $0 < \delta \leq 1$. Here, the convolution $(\gamma^{[i]} * f)(x)$ has to be understood similar to equation (4.8).

- (2) For $0 < \delta \leq \delta_0$, there is a bounded linear map **reconstruction map** $R^{(\delta)} : \mathcal{C}_{p,q,s,\mu}^{(\alpha)} \rightarrow M_{(s,\mu),\alpha}^{p,q}(\mathbb{R}^d)$ satisfying $R^{(\delta)} \circ A^{(\delta)} = \text{id}_{M_{(s,\mu),\alpha}^{p,q}(\mathbb{R}^d)}$. Furthermore, the action of $R^{(\delta)}$ on a given sequence is independent of the precise choice of p, q, s, μ .
- (3) We have the following **consistency statement**: If $f \in M_{(s,\mu),\alpha}^{p,q}(\mathbb{R}^d)$ and if $p_0 \leq \tilde{p} \leq \infty$ and $q_0 \leq \tilde{q} \leq \infty$ and if furthermore $|\tilde{s}| \leq s_0$ and $|\tilde{\mu}| \leq \mu_0$, then the following equivalence holds:

$$f \in M_{(\tilde{s},\tilde{\mu}),\alpha}^{\tilde{p},\tilde{q}}(\mathbb{R}^d) \iff A^{(\delta)} f \in \mathcal{C}_{\tilde{p},\tilde{q},\tilde{s},\tilde{\mu}}^{(\alpha)}. \quad \blacktriangleleft$$

Proof. First of all, we remark that it is comparatively easy to show that the family $\Gamma^{(\delta)}$ forms a Banach frame for $M_{(s,\mu),\alpha}^{p,q}(\mathbb{R}^d)$ if $0 < \delta \leq \delta_0$, where δ_0 might depend on p, q, s, μ . About half of the proof will be spent on showing that δ_0 can actually be chosen *independently* of p, q, s, μ , as long as these satisfy the restrictions mentioned in the statement of the theorem.

Recall from Lemma 7.3 that there is a family $\Phi = (\varphi_i)_{i \in \mathbb{Z}^d \setminus \{0\}}$ associated to $\mathcal{Q} = \mathcal{Q}_r^{(\alpha)}$ satisfying Assumption 6.1. Furthermore, Corollary 6.5 yields a function $\varrho \in C_c^\infty(\mathbb{R}^d)$ such that, for $v_0(x) = [2 \cdot (1 + |x|)]^{|\mu|}$ as in Lemma 7.4, with $K = |\mu| \leq \mu_0$ and $Q = B_r(0)$, as well as $p \geq p_0$, we have

$$\begin{aligned} C_{\mathcal{Q}_r^{(\alpha)}, \Phi, v_0, p} &\leq \Omega_0^K \Omega_1 \cdot (4 \cdot d)^{1+2\lceil K + \frac{d+\varepsilon}{p} \rceil} \cdot \left(\frac{s_d}{\varepsilon} \right)^{1/p} \cdot 2^{\lceil K + \frac{d+\varepsilon}{p} \rceil} \cdot \lambda_d(Q) \cdot \max_{|\beta| \leq \lceil K + \frac{d+\varepsilon}{p} \rceil} \|\partial^\beta \varrho\|_{\sup} \cdot \max_{|\beta| \leq \lceil K + \frac{d+\varepsilon}{p} \rceil} C^{(\beta)} \\ &\leq 2^{\mu_0} \lambda_d(Q) \cdot (8 \cdot d)^{1+2\lceil \mu_0 + \frac{d+\varepsilon}{p_0} \rceil} \cdot \left(1 + \frac{s_d}{\varepsilon} \right)^{\frac{1}{p_0}} \cdot \max_{|\beta| \leq \lceil \mu_0 + \frac{d+\varepsilon}{p_0} \rceil} \|\partial^\beta \varrho\|_{\sup} \cdot \max_{|\beta| \leq \lceil \mu_0 + \frac{d+\varepsilon}{p_0} \rceil} C^{(\beta)} =: L_0. \end{aligned} \quad (7.14)$$

Now, assume that γ satisfies all the stated properties and let $|s| \leq s_0$, $|\mu| \leq \mu_0$ and $p, q \in (0, \infty]$ with $p \geq p_0$ and $q \geq q_0$. We want to verify the assumptions of Corollary 6.6 for the family $(\gamma_i)_{i \in \mathbb{Z}^d \setminus \{0\}}$, with $\gamma_i := \gamma$ for all $i \in \mathbb{Z}^d \setminus \{0\}$ and with $\mathcal{Q} = \mathcal{Q}_r^{(\alpha)}$. To this end, let $\gamma_1^{(0)} := \gamma$ and set $n_i := 1$, so that $\gamma_i = \gamma = \gamma_{n_i}^{(0)}$ for all $i \in \mathbb{Z}^d \setminus \{0\}$. In the notation of Lemma 3.7, we then have $Q^{(1)} = \bigcup \{Q'_i \mid i \in \mathbb{Z}^d \setminus \{0\} \text{ and } n_i = 1\} = B_r(0)$, since $Q_i = T_i[B_r(0)] + b_i$ and thus $Q'_i = B_r(0)$ for all $i \in \mathbb{Z}^d \setminus \{0\}$. In view of part (3) of our assumptions, we thus see that Lemma 3.7 is applicable, so that the family $(\gamma_i)_{i \in \mathbb{Z}^d \setminus \{0\}}$ satisfies Assumption 3.6, with $\Omega_2^{(p,K)} \leq L_1$ for some constant $L_1 = L_1(\gamma, \mu_0, p_0, r, \alpha, d) > 0$, all $p \geq p_0$ and all $K \leq \mu_0$. Recall from Lemma 7.4 that we can choose $K = |\mu| \leq \mu_0$ in our present setting.

Finally, recall from part (4) of our assumptions that there is some $L_2 > 0$ (independent of p, q, s, μ) satisfying $|\partial^\beta \widehat{\gamma}(\xi)| \leq L_2 \cdot (1 + |\xi|)^{-N_0}$ for all $\xi \in \mathbb{R}^d$ and all $\beta \in \mathbb{N}_0^d$ with $|\beta| \leq \left\lceil \mu_0 + \frac{d+\varepsilon}{p_0} \right\rceil$.

With these preparations, we can now verify the prerequisites of Corollary 6.6:

- (1) As we have seen at the beginning of this section, the covering $\mathcal{Q} = \mathcal{Q}_r^{(\alpha)}$, the weight $v = v^{(\mu)}$ (with $v_0(x) = [2 \cdot (1 + |x|)]^{|\mu|}$, $\Omega_0 = 1$ and $\Omega_1 = 2^{|\mu|} \leq 2^{\mu_0}$, as well as $K = |\mu| \leq \mu_0$) satisfy all standing assumptions of Section 1.3. Furthermore, the family Φ satisfies Assumption 6.1.
- (2) By our assumptions, we have $\gamma_i = \gamma \in L_{(1+|\bullet|)^{\mu_0}}^1(\mathbb{R}^d) \hookrightarrow L_{(1+|\bullet|)^K}^1(\mathbb{R}^d)$ and $\widehat{\gamma}_i = \widehat{\gamma} \in C^\infty(\mathbb{R}^d)$ for all $i \in \mathbb{Z}^d \setminus \{0\}$, where all partial derivatives of $\widehat{\gamma}_i = \widehat{\gamma}$ are polynomially bounded.
- (3) By our assumptions, we have $\gamma_i = \gamma \in C^1(\mathbb{R}^d)$ and $\partial_\ell \gamma \in L_{(1+|\bullet|)^{\mu_0}}^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and consequently also $\partial_\ell \gamma_i = \partial_\ell \gamma \in L_{(1+|\bullet|)^K}^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ for all $\ell \in \underline{d}$ and $i \in \mathbb{Z}^d \setminus \{0\}$.
- (4) As seen above, the family $(\gamma_i)_{i \in \mathbb{Z}^d \setminus \{0\}} = (\gamma)_{i \in \mathbb{Z}^d \setminus \{0\}}$ satisfies Assumption 3.6.
- (5) Let

$$C_1 := \sup_{i \in \mathbb{Z}^d \setminus \{0\}} \sum_{j \in \mathbb{Z}^d \setminus \{0\}} M_{j,i} \in [0, \infty] \quad \text{and} \quad C_2 := \sup_{j \in \mathbb{Z}^d \setminus \{0\}} \sum_{i \in \mathbb{Z}^d \setminus \{0\}} M_{j,i} \in [0, \infty]$$

as in Corollary 6.6, i.e., with

$$M_{j,i} := \left(\frac{w_j^{(s^*)}}{w_i^{(s^*)}} \right)^\tau \cdot (1 + \|T_j^{-1} T_i\|)^\sigma \cdot \max_{|\beta| \leq 1} \left(|\det T_i|^{-1} \cdot \int_{Q_i} \max_{|\theta| \leq N} |(\partial^\theta \widehat{\partial^\beta \gamma})(S_j^{-1} \xi)| \, d\xi \right)^\tau,$$

where, $s^* = \frac{s}{1-\alpha}$ and, since $K = |\mu|$,

$$\begin{aligned} N &= \left\lceil |\mu| + \frac{d+\varepsilon}{\min\{1, p\}} \right\rceil, \\ \tau &= \min\{1, p, q\}, \\ \sigma &= \tau \cdot \left(\frac{d}{\min\{1, p\}} + |\mu| + \left\lceil |\mu| + \frac{d+\varepsilon}{\min\{1, p\}} \right\rceil \right). \end{aligned}$$

Note $\min\{1, p\} \geq \min\{1, p_0\} = p_0$ and $|\mu| \leq \mu_0$, so that $N \leq \left\lceil \mu_0 + \frac{d+\varepsilon}{p_0} \right\rceil$. Hence, we see that equation (7.4) (with $C = L_2$) and hence also equation (7.5) is satisfied, i.e., we have

$$M_{j,i} \leq (2^{N+1} \pi \cdot L_2)^\tau \cdot M_{j,i}^{(0)} \quad \forall i, j \in \mathbb{Z}^d \setminus \{0\},$$

with $M_{j,i}^{(0)}$ as in equation (7.5). We now want to apply Lemma 7.6. To this end, we have to verify that

$$N_0 \geq d + 2 + \frac{d+1}{\tau} + \max \left\{ |s^* + d\alpha_0|, \left| s^* + \left(d - \frac{\sigma}{\tau} \right) \alpha_0 \right| \right\}.$$

But we have $\tau = \min\{1, p, q\} \geq \min\{1, p_0, q_0\} = \min\{p_0, q_0\} =: \tau_0$. Furthermore, with $s_0^* := s_0 / (1 - \alpha)$, we have $|s^* + d\alpha_0| \leq s_0^* + d\alpha_0 = \frac{1}{1-\alpha} (s_0 + \alpha d)$ and

$$\frac{\sigma}{\tau} = \frac{d}{\min\{1, p\}} + |\mu| + \left\lceil |\mu| + \frac{d+\varepsilon}{\min\{1, p\}} \right\rceil \geq \frac{d}{\min\{1, p\}} \geq d,$$

so that

$$\begin{aligned} \left| s^* + \left(d - \frac{\sigma}{\tau} \right) \alpha_0 \right| &\leq s_0^* + \alpha_0 \left| d - \frac{\sigma}{\tau} \right| = s_0^* + \alpha_0 \left(\frac{\sigma}{\tau} - d \right) \\ &= \frac{1}{1-\alpha} \left[s_0 + \alpha \left(\frac{d}{\min\{1, p\}} + |\mu| + \left\lceil |\mu| + \frac{d+\varepsilon}{\min\{1, p\}} \right\rceil - d \right) \right] \\ &\leq \frac{1}{1-\alpha} \left[s_0 + \alpha \left(\frac{d}{p_0} - d + \mu_0 + \left\lceil \mu_0 + \frac{d+\varepsilon}{p_0} \right\rceil \right) \right]. \end{aligned}$$

Hence, our assumptions easily yield

$$\begin{aligned} N_0 &= d + 2 + \frac{d+1}{\tau_0} + \frac{1}{1-\alpha} \cdot \max \left\{ s_0 + \alpha d, s_0 + \alpha \left(\frac{d}{p_0} - d + \mu_0 + \left\lceil \mu_0 + \frac{d+\varepsilon}{p_0} \right\rceil \right) \right\} \\ &\geq d + 2 + \frac{d+1}{\tau} + \max \left\{ |s^* + d\alpha_0|, \left| s^* + \left(d - \frac{\sigma}{\tau} \right) \alpha_0 \right| \right\}, \end{aligned}$$

as desired.

For brevity, set $L_3 := 2^d \cdot (1 + r \cdot (2 + 4r)^{\alpha_0})^{N_0+d}$. Since Lemma 7.6 is applicable, and since $\tau_0 \leq \tau \leq 1$, we get

$$\begin{aligned} C_1^{1/\tau} &\leq 2^{N+1} \pi \cdot L_2 \cdot \left[\sup_{i \in \mathbb{Z}^d \setminus \{0\}} \sum_{j \in \mathbb{Z}^d \setminus \{0\}} M_{j,i}^{(0)} \right]^{1/\tau} \\ &\leq 2^{N+1} \pi L_2 \cdot 2^{|s^*| + \frac{1+\sigma}{\tau}} 6^{\frac{d}{\tau}} \cdot \max \left\{ 4^{\alpha_0(\frac{\sigma}{\tau} + d) + |s^*|} \cdot 12^{N_0} \cdot s_d, (2 + 4r)^{|s^*| + \alpha_0[\frac{\sigma}{\tau} + d + N_0]} L_3 \right\} \\ &\stackrel{(*)}{\leq} 6^{\frac{d}{\tau_0}} 2^{1 + \lceil \mu_0 + \frac{d+\varepsilon}{p_0} \rceil} \pi L_2 \cdot 2^{\frac{s_0}{1-\alpha} + \frac{1}{\tau_0} + \frac{\sigma_0}{\tau_0}} \cdot \max \left\{ 4^{\frac{1}{1-\alpha} [s_0 + \alpha(\frac{\sigma_0}{\tau_0} + d)]} \cdot 12^{N_0} s_d, (2 + 4r)^{\frac{1}{1-\alpha} [s_0 + \alpha(\frac{\sigma_0}{\tau_0} + d + N_0)]} L_3 \right\} \\ &=: L_4. \end{aligned}$$

Here, we recall $\tau_0 = \min \{p_0, q_0\}$ and observe that the step marked with $(*)$ used the estimate

$$\frac{\sigma}{\tau} = \frac{d}{\min \{1, p\}} + |\mu| + \left| |\mu| + \frac{d + \varepsilon}{\min \{1, p\}} \right| \leq \frac{d}{p_0} + \mu_0 + \left\lceil \mu_0 + \frac{d + \varepsilon}{p_0} \right\rceil =: \frac{\sigma_0}{\tau_0}$$

and $\sigma = \frac{\sigma}{\tau} \cdot \tau \leq \frac{\sigma}{\tau} \leq \frac{\sigma_0}{\tau_0}$. Hence, we have shown $C_1^{1/\tau} \leq L_4$, where L_4 is independent of p, q, s, μ . Exactly the same argument also shows $C_2^{1/\tau} \leq L_4$, with the same constant L_4 . In particular, $C_1 < \infty$ and $C_2 < \infty$, which was the last part of Corollary 6.6 that we needed to verify.

All in all, Corollary 6.6 shows that the family $(\gamma_i)_{i \in \mathbb{Z}^d \setminus \{0\}} = (\gamma)_{i \in \mathbb{Z}^d \setminus \{0\}}$ satisfies all assumptions of Theorem 4.7 and also that $\|\vec{A}\|^{\max\{1, \frac{1}{p}\}} \leq 2L_5^{(0)} \cdot L_4$, as well as $\|\vec{B}\|^{\max\{1, \frac{1}{p}\}} \leq 2L_5^{(0)} \cdot L_4$, where \vec{A} and \vec{B} are defined as in Assumptions 3.1 and 4.1 and where

$$\begin{aligned} L_5^{(0)} &= \Omega_0^K \Omega_1 \cdot d^{1/\min\{1, p\}} \cdot (4 \cdot d)^{1+2\lceil K + \frac{d+\varepsilon}{\min\{1, p\}} \rceil} \cdot \left(\frac{s_d}{\varepsilon} \right)^{1/\min\{1, p\}} \cdot \max_{|\beta| \leq \lceil K + \frac{d+\varepsilon}{\min\{1, p\}} \rceil} C^{(\beta)} \\ &\leq 2^{\mu_0} \cdot d^{1/p_0} \cdot (4 \cdot d)^{1+2\lceil \mu_0 + \frac{d+\varepsilon}{p_0} \rceil} \cdot \left[1 + \frac{s_d}{\varepsilon} \right]^{\frac{1}{p_0}} \cdot \max_{|\beta| \leq \lceil \mu_0 + \frac{d+\varepsilon}{p_0} \rceil} C^{(\beta)} =: L_5, \end{aligned}$$

where the constants $C^{(\beta)} = C^{(\beta)}(\Phi)$ are defined as in Assumption 6.1.

Now, Theorem 4.7 shows that the family $\Gamma^{(\delta)}$ is a Banach frame for $M_{(s, \mu), \alpha}^{p, q}(\mathbb{R}^d) = \mathcal{D}(\mathcal{Q}_r^{(\alpha)}, L_{v(\mu)}^p, \ell_{w(s^*)}^q)$, as soon as $0 < \delta \leq \frac{1}{1+2\|\vec{F}_0\|^2}$ with F_0 as in Lemma 4.6. But that lemma yields the estimate

$$\begin{aligned} \|\vec{F}_0\| &\leq 2^{\frac{1}{q}} C_{\mathcal{Q}_r^{(\alpha)}, \Phi, v_0, p}^2 \cdot \|\Gamma_{\mathcal{Q}_r^{(\alpha)}}\|^2 \cdot \left(\|\vec{A}\|^{\max\{1, \frac{1}{p}\}} + \|\vec{B}\|^{\max\{1, \frac{1}{p}\}} \right) \cdot L_6^{(0)} \\ (\text{eqs. (1.15) and (7.14)}) &\leq 2^{\frac{1}{q_0}} \cdot C_{\mathcal{Q}_r^{(\alpha)}, w(s^*)}^2 \cdot N_{\mathcal{Q}_r^{(\alpha)}}^{2(1+\frac{1}{q})} \cdot 4L_0^2 L_4 L_5 \cdot L_6^{(0)} \end{aligned}$$

for

$$L_6^{(0)} = \begin{cases} \frac{(2^{16} \cdot 768 / d^{\frac{3}{2}})^{\frac{d}{p}}}{2^{42} \cdot 12^d \cdot d^{15}} \cdot \left(2^{52} \cdot d^{\frac{25}{2}} \cdot N^3 \right)^{N+1} \cdot N_{\mathcal{Q}_r^{(\alpha)}}^{2(\frac{1}{p}-1)} \left(1 + R_{\mathcal{Q}_r^{(\alpha)}} C_{\mathcal{Q}_r^{(\alpha)}} \right)^{d(\frac{1}{p}-1)} \cdot \Omega_0^{13K} \Omega_1^{13} \Omega_2^{(p, K)}, & \text{if } p < 1, \\ \frac{1}{\sqrt{d} \cdot 2^{12+6\lceil K \rceil}} \cdot (2^{17} \cdot d^{5/2} \cdot N)^{\lceil K \rceil + d+2} \cdot \left(1 + R_{\mathcal{Q}_r^{(\alpha)}} \right)^d \cdot \Omega_0^{3K} \Omega_1^3 \Omega_2^{(p, K)}, & \text{if } p \geq 1. \end{cases}$$

But above we saw $\Omega_2^{(p, K)} \leq L_1$ since $K = |\mu| \leq \mu_0$ and $p \geq p_0$. Using this estimate and the bounds $N \leq \left\lceil \mu_0 + \frac{d+\varepsilon}{p_0} \right\rceil$ and $0 \leq K = |\mu| \leq \mu_0$, as well as $\Omega_0 = 1$ and $\Omega_1 = 2^{|\mu|} \leq 2^{\mu_0}$, we see $L_6^{(0)} \leq L_6$, where L_6 is independent of p, q, s, μ .

Finally, it is not hard to see—because of $w(s^*) = (w^{(1/(1-\alpha))})^s$ —that

$$N_{\mathcal{Q}_r^{(\alpha)}, w(s^*)} \leq N_{\mathcal{Q}_r^{(\alpha)}, w(1/(1-\alpha))}^{|s|} \leq N_{\mathcal{Q}_r^{(\alpha)}, w(1/(1-\alpha))}^{s_0} =: L_7,$$

where $L_7 > 0$ is independent of p, q, s, μ . By putting everything together, we get $\|\vec{F}_0\| \leq L_8$, with L_8 independent of p, q, s, μ . Hence, $\Gamma^{(\delta)}$ is a Banach frame for $M_{(s, \mu), \alpha}^{p, q}(\mathbb{R}^d)$ in the sense of Theorem 4.7 as soon as $0 < \delta \leq \delta_0 := \frac{1}{1+2L_8^2}$, with δ_0 independent of p, q, s, μ .

All that remains to verify is that the space $\ell_{\left(\det T_i\right)^{\frac{1}{2}-\frac{1}{p}} \cdot w_i}^q \left([C_i^{(\delta)}]_{i \in I} \right)$ (with $w_i = w_i^{(s^*)} = \langle i \rangle^{s^*}$) appearing in

Theorem 4.7 coincides with the space $\mathcal{C}_{p, q, s, \mu}^{(\alpha)}$ from the statement of the current theorem. To this end, first note

that $|i| \asymp \langle i \rangle$ for all $i \in \mathbb{Z}^d \setminus \{0\}$, so that

$$|\det T_i|^{\frac{1}{2}-\frac{1}{p}} \cdot w_i = |i|^{d\alpha_0(\frac{1}{2}-\frac{1}{p})} \cdot \langle i \rangle^{s^*} \asymp_s |i|^{\frac{1}{1-\alpha}[s+\alpha d(\frac{1}{2}-\frac{1}{p})]} \quad \forall i \in \mathbb{Z}^d \setminus \{0\}.$$

Finally, recall from equation (4.2) that

$$C_i^{(\delta)} = \ell_{v^{(j,\delta)}}^p(\mathbb{Z}^d) \quad \text{with} \quad v_k^{(j,\delta)} = v(\delta \cdot T_j^{-T} k)$$

where $v = v^{(\mu)}$ with $v^{(\mu)}(x) = \langle x \rangle^\mu \asymp_\mu (1 + |x|)^\mu$ for all $x \in \mathbb{R}^d$. Furthermore, since $0 < \delta \leq 1$, we have

$$\delta \cdot (1 + |x|) \leq 1 + |\delta \cdot x| \leq 1 + |x| \quad \forall x \in \mathbb{R}^d,$$

which yields

$$v_k^{(j,\delta)} = v(\delta \cdot T_j^{-T} k) \asymp_\mu (1 + |\delta \cdot T_j^{-T} k|)^\mu \asymp_{\mu,\delta} (1 + |T_j^{-T} k|)^\mu = (1 + |k|/|j|^{\alpha_0})^\mu$$

for all $k \in \mathbb{Z}^d$ and $j \in \mathbb{Z}^d \setminus \{0\}$. Here, the implied constant might depend on μ, δ , but not on j, k . Combining these facts, we conclude $\ell_{\left(|\det T_i|^{\frac{1}{2}-\frac{1}{p}} \cdot w_i\right)_{i \in I}}^q \left([C_i^{(\delta)}]_{i \in I}\right) = \mathcal{C}_{p,q,s,\mu}^{(\alpha)}$, with equivalent quasi-norms, as desired. Now all claims follow from Theorem 4.7. \square

Having established convenient criteria for the existence of Banach frames, we finally consider nice criteria which ensure that a prototype γ generates an atomic decomposition for a given α -modulation space.

Theorem 7.8. Let $d \in \mathbb{N}$, $\alpha \in [0, 1)$ and choose $r > r_1(d, \alpha)$ with $r_1(d, \alpha)$ as in Theorem 7.1.

Let $s_0, \mu_0 \geq 0$ and $p_0, q_0 \in (0, 1]$, as well as $\varepsilon \in (0, 1)$. Assume that $\gamma : \mathbb{R}^d \rightarrow \mathbb{C}$ is measurable and satisfies the following conditions:

- (1) We have $\|\gamma\|_{\mu_0 + \frac{d}{p_0} + 1} < \infty$, where as usual $\|g\|_M = \sup_{x \in \mathbb{R}^d} (1 + |x|)^M |g(x)|$. In particular, $\gamma \in L^1(\mathbb{R}^d)$.
- (2) We have $\hat{\gamma} \in C^\infty(\mathbb{R}^d)$ and all partial derivatives of $\hat{\gamma}$ are polynomially bounded.
- (3) We have $|\hat{\gamma}(\xi)| \geq c > 0$ for all $\xi \in \overline{B_r}(0)$.
- (4) We have

$$|(\partial^\beta \hat{\gamma})(\xi)| \lesssim (1 + |\xi|)^{-M_0}$$

for all $\xi \in \mathbb{R}^d$ and all $\beta \in \mathbb{N}_0^d$ with $|\beta| \leq \left\lceil \mu_0 + \frac{d+\varepsilon}{p_0} \right\rceil$, where

$$M_0 = (d+1) \cdot \left(2 + \varepsilon + \frac{1}{\min\{p_0, q_0\}} \right) + \Lambda,$$

with

$$\Lambda = \begin{cases} \frac{1}{1-\alpha} \max\{s_0 + d\alpha, s_0 + \alpha(\lceil \mu_0 + d + \varepsilon \rceil - d)\}, & \text{if } p_0 = 1, \\ \frac{1}{1-\alpha} [s_0 + \alpha(\mu_0 + \lceil \mu_0 + p_0^{-1} \cdot (d + \varepsilon) \rceil)], & \text{if } p_0 \in (0, 1). \end{cases}$$

Then there is some $\delta_0 > 0$ such that for all $0 < \delta \leq \delta_0$, the family

$$\Gamma^{(\delta)} := \left(L_{\delta \cdot k/|i|^{\alpha_0}} \gamma^{[i]} \right)_{i \in \mathbb{Z}^d \setminus \{0\}, k \in \mathbb{Z}^d}, \quad \text{with} \quad \gamma^{[i]} = |i|^{\frac{d-\alpha_0}{2}} \cdot M_{|i|^{\alpha_0}, i} [\gamma \circ |i|^{\alpha_0} \text{id}]$$

forms an atomic decomposition for $M_{(s,\mu),\alpha}^{p,q}(\mathbb{R}^d)$ for all $|s| \leq s_0$, $|\mu| \leq \mu_0$ and all $p, q \in (0, \infty]$ with $p \geq p_0$ and $q \geq q_0$.

Precisely, this means the following: Define the coefficient space

$$\mathcal{C}_{p,q,s,\mu}^{(\alpha)} := \ell_{\left[|i|^{\frac{1}{1-\alpha}(s+\alpha d(\frac{1}{2}-\frac{1}{p}))}\right]_{i \in \mathbb{Z}^d \setminus \{0\}}}^q \left(\left[\ell_{\left[(1+|k|/|i|^{\alpha_0})^\mu\right]_{k \in \mathbb{Z}^d}}^p(\mathbb{Z}^d) \right]_{i \in \mathbb{Z}^d \setminus \{0\}} \right).$$

Then the following hold:

- (1) For arbitrary $\delta \in (0, 1]$, the **synthesis map**

$$S^{(\delta)} : \mathcal{C}_{p,q,s,\mu}^{(\alpha)} \rightarrow M_{(s,\mu),\alpha}^{p,q}(\mathbb{R}^d), (c_k^{(i)})_{i \in \mathbb{Z}^d \setminus \{0\}, k \in \mathbb{Z}^d} \mapsto \sum_{i \in \mathbb{Z}^d \setminus \{0\}} \sum_{k \in \mathbb{Z}^d} \left[c_k^{(i)} \cdot L_{\delta \cdot k/|i|^{\alpha_0}} \gamma^{[i]} \right]$$

is well-defined and bounded. Convergence of the series has to be understood as described in the remark following Theorem 5.6.

- (2) For $0 < \delta \leq \delta_0$, there is a bounded linear **coefficient map** $C^{(\delta)} : M_{(s,\mu),\alpha}^{p,q}(\mathbb{R}^d) \rightarrow \mathcal{C}_{p,q,s,\mu}^{(\alpha)}$ satisfying $S^{(\delta)} \circ C^{(\delta)} = \text{id}_{M_{(s,\mu),\alpha}^{p,q}(\mathbb{R}^d)}$. Furthermore, the action of $C^{(\delta)}$ on a given $f \in M_{(s,\mu),\alpha}^{p,q}(\mathbb{R}^d)$ is independent of the precise choice of p, q, s, μ . \blacktriangleleft

Remark. Choose M_0 as in the theorem above. If $\gamma \in C_c^{[M_0]}(\mathbb{R}^d)$, then $\widehat{\gamma} \in C^\infty(\mathbb{R}^d)$ with all partial derivatives being polynomially bounded. Furthermore, for arbitrary $\alpha \in \mathbb{N}_0^d$, we have $\gamma_\alpha \in C_c^{[M_0]}(\mathbb{R}^d) \hookrightarrow W^{[M_0],1}(\mathbb{R}^d)$ for

$$\gamma_\alpha : \mathbb{R}^d \rightarrow \mathbb{C}, x \mapsto (-2\pi i x)^\alpha \cdot \gamma(x).$$

But by differentiation under the integral, it is not hard to see $\partial^\alpha \widehat{\gamma}(\xi) = \widehat{\gamma_\alpha}(\xi)$ for all $\xi \in \mathbb{R}^d$. Hence, Lemma 6.3 yields $|\partial^\alpha \widehat{\gamma}(\xi)| = |(\mathcal{F}^{-1}\gamma_\alpha)(-\xi)| \lesssim (1 + |\xi|)^{-[M_0]} \leq (1 + |\xi|)^{-M_0}$. Finally, we clearly have $\|\gamma\|_{\mu_0 + \frac{d}{p_0} + 1} < \infty$, since γ has compact support.

All in all, these considerations show that every prototype $\gamma \in C_c^{[M_0]}(\mathbb{R}^d)$ with $\widehat{\gamma}(\xi) \neq 0$ for all $\xi \in \overline{B_r}(0)$ generates an atomic decomposition for $M_{(s,\mu),\alpha}^{p,q}(\mathbb{R}^d)$, where $M_0 = M_0(d, p, q, s, \mu, \alpha, \varepsilon)$ has to be chosen suitably. Very similar considerations apply for the case of Banach frames: Here, it suffices to have $\gamma \in C_c^{[N_0]}(\mathbb{R}^d)$ with $\widehat{\gamma}(\xi) \neq 0$ for all $\xi \in \overline{B_r}(0)$, where $N_0 = N_0(d, p, q, s, \mu, \alpha, \varepsilon)$ is chosen as in Theorem 7.7. \blacklozenge

Proof. First of all, we remark as in the proof of Theorem 7.7 that it is comparatively easy to show that the family $\Gamma^{(\delta)}$ forms an atomic decomposition for $M_{(s,\mu),\alpha}^{p,q}(\mathbb{R}^d)$ if $0 < \delta \leq \delta_0$, where δ_0 might depend on p, q, s, μ . About half of the proof will be spent on showing that δ_0 can actually be chosen *independently* of p, q, s, μ , as long as these satisfy the restrictions mentioned in the statement of the theorem.

Our assumptions yield $L_0 > 0$ satisfying $|\partial^\beta \widehat{\gamma}(\xi)| \leq L_0 \cdot (1 + |\xi|)^{-M_0}$ for all $\xi \in \mathbb{R}^d$ and all multiindices $\beta \in \mathbb{N}_0^d$ with $|\beta| \leq \left\lceil \mu_0 + \frac{d+\varepsilon}{p_0} \right\rceil =: N_0$. As our first step, we invoke Lemma 6.9 with $N = N_0$ and

$$\varrho : \mathbb{R}^d \rightarrow (0, \infty), \xi \mapsto L_0 \cdot (1 + |\xi|)^{-[M_0 - (d+1)(1+\varepsilon)]}.$$

To this end, we observe $N_0 \geq \lceil (d + \varepsilon)/p_0 \rceil \geq \lceil d + \varepsilon \rceil = d + 1$ and furthermore

$$M_0 - (d + 1)(1 + \varepsilon) \geq (d + 1) \cdot (2 + \varepsilon) - (d + 1)(1 + \varepsilon) = d + 1 > d,$$

so that $\varrho \in L^1(\mathbb{R}^d)$. Finally, as we just saw, we indeed have

$$|\partial^\beta \widehat{\gamma}(\xi)| \leq L_0 \cdot (1 + |\xi|)^{-M_0} = \varrho(\xi) \cdot (1 + |\xi|)^{-(d+1)(1+\varepsilon)} \leq \varrho(\xi) \cdot (1 + |\xi|)^{-(d+1+\varepsilon)}$$

for all $\xi \in \mathbb{R}^d$ and all $\beta \in \mathbb{N}_0^d$ with $|\beta| \leq N_0$, so that all assumptions of Lemma 6.9 are satisfied. Hence, there are functions $\gamma_1, \gamma_2 \in L^1(\mathbb{R}^d)$ with the following properties:

- (1) We have $\gamma = \gamma_1 * \gamma_2$.
- (2) We have $\gamma_2 \in C^1(\mathbb{R}^d)$ with $L_1^{(M)} := \|\gamma_2\|_M + \|\nabla \gamma_2\|_M < \infty$ for arbitrary $M \in \mathbb{N}_0$.
- (3) We have $\widehat{\gamma}_1, \widehat{\gamma}_2 \in C^\infty(\mathbb{R}^d)$, where all partial derivatives of these functions are polynomially bounded.
- (4) We have $\|\gamma_1\|_{N_0} < \infty$ and $\|\gamma\|_{N_0} < \infty$.
- (5) We have $|\partial^\beta \widehat{\gamma}_1(\xi)| \leq L_2 \cdot \varrho(\xi) = L_0 L_2 \cdot (1 + |\xi|)^{-M_{00}}$ for all $\xi \in \mathbb{R}^d$ and all $\beta \in \mathbb{N}_0^d$ with $|\beta| \leq N_0$. Here, $L_2 := 2^{1+d+4N_0} \cdot N_0! \cdot (1 + d)^{N_0}$ and

$$M_{00} := M_0 - (d + 1)(1 + \varepsilon) = (d + 1) \cdot \left(1 + \frac{1}{\min\{p_0, q_0\}}\right) + \Lambda.$$

Next, recall from Lemma 7.3 that there is a family $\Phi = (\varphi_i)_{i \in \mathbb{Z}^d \setminus \{0\}}$ associated to $\mathcal{Q} = \mathcal{Q}_r^{(\alpha)}$ satisfying Assumption 6.1. As in the proof of Theorem 7.7 (cf. equation (7.14)), we get as a consequence of Corollary 6.5 a constant $L_3 > 0$ satisfying

$$C_{\mathcal{Q}_r^{(\alpha)}, \Phi, v_0, p} \leq L_3, \quad \forall p \geq p_0 \forall K = |\mu| \leq \mu_0, \quad (7.15)$$

where $v_0(x) = [2 \cdot (1 + |x|)]^{|\mu|}$ is as in Lemma 7.4.

Now, let p, q, s, μ as in the statement of the Theorem. We want to show that the family $(\gamma_i)_{i \in \mathbb{Z}^d \setminus \{0\}}$ with $\gamma_i := \gamma$ for all $i \in \mathbb{Z}^d \setminus \{0\}$ satisfies all assumptions of Corollary 6.7, for $\mathcal{Q} = \mathcal{Q}_r^{(\alpha)}$. To this end, let $\gamma_1^{(0)} := \gamma$ and $n_i := 1$, so that $\gamma_i = \gamma = \gamma_{n_i}^{(0)}$ for all $i \in \mathbb{Z}^d \setminus \{0\}$. As in the proof of Theorem 7.7, we then see that all assumptions of Lemma 3.7 are satisfied. Hence, the family $(\gamma_i)_{i \in \mathbb{Z}^d \setminus \{0\}}$ satisfies Assumption 3.6 and we also get $\Omega_2^{(p,K)} \leq L_4$ for some constant $L_4 = L_4(\gamma, \mu_0, p_0, r, \alpha, d) > 0$, all $p \geq p_0$ and all $K \leq \mu_0$.

Set $\gamma_{i,1} := \gamma_1$ and $\gamma_{i,2} := \gamma_2$ for $i \in \mathbb{Z}^d \setminus \{0\}$. Let us verify the assumptions of Corollary 6.7 for these choices:

- (1) As we have seen at the beginning of this section, the covering $\mathcal{Q} = \mathcal{Q}_r^{(\alpha)}$, the weight $v = v^{(\mu)}$ (with $v_0(x) = [2 \cdot (1 + |x|)]^{|\mu|}$, $\Omega_0 = 1$ and $\Omega_1 = 2^{|\mu|} \leq 2^{\mu_0}$, as well as $K = |\mu| \leq \mu_0$) satisfy all standing assumptions of Section 1.3. Furthermore, the family $\Phi = (\varphi_i)_{i \in I}$ from above satisfies Assumption 6.1.
- (2) Our choice of γ_1, γ_2 from above ensures that all $\gamma_i = \gamma$, $\gamma_{i,1} = \gamma_1$ and $\gamma_{i,2} = \gamma_2$ are measurable functions.

- (3) As seen above, we have $\|\gamma_1\|_{N_0} < \infty$. Since $N_0 = \left\lceil \mu_0 + \frac{d+\varepsilon}{p_0} \right\rceil \geq \mu_0 + \frac{d+\varepsilon}{p_0} \geq \mu_0 + d + \varepsilon$, equation (1.9) easily yields $\gamma_{i,1} = \gamma_1 \in L^1_{(1+|\bullet|)^{\mu_0}}(\mathbb{R}^d) \hookrightarrow L^1_{(1+|\bullet|)^K}(\mathbb{R}^d)$ for all $i \in \mathbb{Z}^d \setminus \{0\}$.
- (4) As seen above, we have $\gamma_{i,2} = \gamma_2 \in C^1(\mathbb{R}^d)$ for all $i \in \mathbb{Z}^d \setminus \{0\}$.
- (5) With $K_0 := K + \frac{d}{\min\{1,p\}} + 1$, we have

$$\begin{aligned} \Omega_4^{(p,K)} &= \sup_{i \in \mathbb{Z}^d \setminus \{0\}} \|\gamma_{i,2}\|_{K_0} + \sup_{i \in \mathbb{Z}^d \setminus \{0\}} \|\nabla \gamma_{i,2}\|_{K_0} \\ &\quad \left(\text{since } \gamma_{i,2} = \gamma_2 \text{ and } K_0 \leq \mu_0 + \frac{d}{p_0} + 1 \right) \leq 2 \cdot \sup_{i \in \mathbb{Z}^d \setminus \{0\}} \left(\|\gamma_2\|_{\lceil \mu_0 + \frac{d}{p_0} + 1 \rceil} + \|\nabla \gamma_2\|_{\lceil \mu_0 + \frac{d}{p_0} + 1 \rceil} \right) \\ &= 2 \cdot L_1^{\left(\lceil \mu_0 + \frac{d}{p_0} + 1 \rceil\right)} =: L_5 < \infty. \end{aligned}$$

- (6) By our assumptions, we have $\|\gamma_i\|_{K_0} = \|\gamma\|_{K_0} \leq \|\gamma\|_{\mu_0 + \frac{d}{p_0} + 1} < \infty$ for all $i \in \mathbb{Z}^d \setminus \{0\}$.
- (7) By choice of γ_1, γ_2 , we have $\gamma_i = \gamma = \gamma_1 * \gamma_2 = \gamma_{i,1} * \gamma_{i,2}$ for all $i \in \mathbb{Z}^d \setminus \{0\}$.
- (8) By the properties of γ_1, γ_2 from above, we have $\widehat{\gamma_{i,\ell}} = \widehat{\gamma}_\ell$ and all partial derivatives of this function are polynomially bounded, for arbitrary $i \in \mathbb{Z}^d \setminus \{0\}$ and $\ell \in \{1, 2\}$.
- (9) As seen above, the family $(\gamma_i)_{i \in \mathbb{Z}^d \setminus \{0\}}$ satisfies Assumption 3.6.
- (10) Let

$$K_1 := \sup_{i \in \mathbb{Z}^d \setminus \{0\}} \sum_{j \in \mathbb{Z}^d \setminus \{0\}} N_{i,j} \in [0, \infty] \quad \text{and} \quad K_2 := \sup_{j \in \mathbb{Z}^d \setminus \{0\}} \sum_{i \in \mathbb{Z}^d \setminus \{0\}} N_{i,j} \in [0, \infty]$$

as in Corollary 6.7, i.e., with

$$N_{i,j} = \left[\frac{w_i^{(s^*)}}{w_j^{(s^*)}} \cdot (|\det T_j| / |\det T_i|)^\vartheta \right]^\tau \cdot (1 + \|T_j^{-1} T_i\|)^\sigma \cdot \left(|\det T_i|^{-1} \cdot \int_{Q_i} \max_{|\beta| \leq N} |(\partial^\beta \widehat{\gamma}_1)(S_j^{-1} \xi)| d\xi \right)^\tau,$$

where $s^* = \frac{s}{1-\alpha}$ and, since $K = |\mu|$,

$$\begin{aligned} N &= \left\lceil |\mu| + \frac{d+\varepsilon}{\min\{1,p\}} \right\rceil, \\ \tau &= \min\{1, p, q\}, \\ \vartheta &= \begin{cases} 0, & \text{if } p \in [1, \infty], \\ \frac{1}{p} - 1, & \text{if } p \in (0, 1), \end{cases} \\ \sigma &= \begin{cases} \tau \cdot \lceil |\mu| + d + \varepsilon \rceil, & \text{if } p \in [1, \infty], \\ \tau \cdot \left(\frac{d}{p} + |\mu| + \lceil |\mu| + \frac{d+\varepsilon}{p} \rceil \right), & \text{if } p \in (0, 1). \end{cases} \end{aligned}$$

Note that $\vartheta \geq 0$ and $\tau > 0$. Furthermore, since $\alpha_0 \geq 0$ and since $|j| \geq 1$ for all $j \in \mathbb{Z}^d \setminus \{0\}$, we have

$$(2\langle j \rangle)^{d\alpha_0} \geq \langle j \rangle^{d\alpha_0} \geq |\det T_j| = |j|^{d\alpha_0} \geq \left(\frac{1}{2} \langle j \rangle \right)^{d\alpha_0}$$

and thus

$$\left[\frac{w_i^{(s^*)}}{w_j^{(s^*)}} \cdot \left(\frac{|\det T_j|}{|\det T_i|} \right)^\vartheta \right]^\tau \leq \left[\frac{w_i^{(s^*)}}{w_j^{(s^*)}} \cdot \left(\frac{2\langle j \rangle}{\frac{1}{2}\langle i \rangle} \right)^{\vartheta d\alpha_0} \right]^\tau \leq 4^{\tau \vartheta d\alpha_0} \cdot \left[\frac{w_i^{(s^* - \vartheta d\alpha_0)}}{w_j^{(s^* - \vartheta d\alpha_0)}} \right]^\tau = 4^{\tau \vartheta d\alpha_0} \cdot \left[\frac{w_j^{(\vartheta d\alpha_0 - s^*)}}{w_i^{(\vartheta d\alpha_0 - s^*)}} \right]^\tau$$

for all $i, j \in \mathbb{Z}^d \setminus \{0\}$. Moreover, $N \leq \left\lceil \mu_0 + \frac{d+\varepsilon}{p_0} \right\rceil = N_0$ and hence $|(\partial^\beta \widehat{\gamma}_1)(\xi)| \leq L_0 L_2 \cdot (1 + |\xi|)^{-M_{00}}$ for all $|\beta| \leq N$. Combining these estimates and noting $\vartheta \leq \frac{1}{p_0}$, we arrive at

$$N_{i,j} \leq \left(L_0 L_2 \cdot 4^{\frac{d\alpha_0}{p_0}} \right)^\tau \cdot \left[\frac{w_j^{(\vartheta d\alpha_0 - s^*)}}{w_i^{(\vartheta d\alpha_0 - s^*)}} \right]^\tau \cdot (1 + \|T_j^{-1} T_i\|)^\sigma \cdot \left(|\det T_i|^{-1} \cdot \int_{Q_i} (1 + |S_j^{-1} \xi|)^{-M_{00}} d\xi \right)^\tau$$

for all $i, j \in \mathbb{Z}^d \setminus \{0\}$. For brevity, set $L_6 := \left(L_0 L_2 \cdot 4^{\frac{d\alpha_0}{p_0}} \right)^\tau$. Note that with this notation, the preceding estimate shows $N_{i,j} \leq N_6 \cdot M_{j,i}^{(0)}$, where $M_{j,i}^{(0)}$ is defined as in equation (7.5), but with s^* replaced by $s^\natural := \vartheta d\alpha_0 - s^*$ and with N_0 replaced by $M_{00} + 1$.

Now, we want to apply Lemma 7.6 to estimate $M_{j,i}^{(0)}$. To this end, we have to verify

$$\begin{aligned} M_{00} + 1 &\stackrel{!}{\geq} d + 2 + \frac{d+1}{\tau} + \max \left\{ |s^\natural + d\alpha_0|, \left| s^\natural + \left(d - \frac{\sigma}{\tau} \right) \alpha_0 \right| \right\} \\ &= d + 2 + \frac{d+1}{\tau} + \max \left\{ |s^* - d\alpha_0(1 + \vartheta)|, \left| s^* + \left(\frac{\sigma}{\tau} - d(1 + \vartheta) \right) \alpha_0 \right| \right\} \\ &= d + 2 + \frac{d+1}{\tau} + \max \left\{ \left| s^* - \frac{d\alpha_0}{\min\{1, p\}} \right|, \left| s^* + \alpha_0 \left(\frac{\sigma}{\tau} - \frac{d}{\min\{1, p\}} \right) \right| \right\}, \end{aligned}$$

where the last line used that

$$1 + \vartheta = \begin{cases} 1 = \frac{1}{\min\{1, p\}}, & \text{if } p \in [1, \infty], \\ 1 + \frac{1}{p} - 1 = \frac{1}{p} = \frac{1}{\min\{1, p\}}, & \text{if } p \in (0, 1). \end{cases}$$

But we have $\tau = \min\{1, p, q\} \geq \min\{1, p_0, q_0\} = \min\{p_0, q_0\} =: \tau_0$ and furthermore

$$\begin{aligned} \frac{\sigma}{\tau} &= \begin{cases} \lceil |\mu| + d + \varepsilon \rceil, & \text{if } p \in [1, \infty], \\ \frac{d}{p} + |\mu| + \left\lceil |\mu| + \frac{d+\varepsilon}{p} \right\rceil, & \text{if } p \in (0, 1) \end{cases} \\ &\leq \begin{cases} \lceil \mu_0 + d + \varepsilon \rceil, & \text{if } p_0 = 1, \\ \frac{d}{p_0} + \mu_0 + \left\lceil \mu_0 + \frac{d+\varepsilon}{p_0} \right\rceil, & \text{if } p_0 \in (0, 1) \end{cases} \\ &=: \frac{\sigma_0}{\tau_0}. \end{aligned}$$

Hence, in case of $p \in [1, \infty]$, we have

$$|s^* - d\alpha_0(1 + \vartheta)| = \left| s^* - \frac{d\alpha_0}{\min\{1, p\}} \right| \leq \frac{1}{1 - \alpha} (s_0 + \alpha d) \leq \Lambda,$$

as well as

$$\begin{aligned} \left| s^* + \left(\frac{\sigma}{\tau} - d(1 + \vartheta) \right) \alpha_0 \right| &= \left| s^* + \alpha_0 \left(\frac{\sigma}{\tau} - \frac{d}{\min\{1, p\}} \right) \right| \leq \frac{1}{1 - \alpha} (s_0 + \alpha \cdot \lceil |\mu| + d + \varepsilon \rceil - d) \\ &\quad (\text{since } \lceil |\mu| + d + \varepsilon \rceil \geq d) = \frac{1}{1 - \alpha} [s_0 + \alpha \cdot (\lceil |\mu| + d + \varepsilon \rceil - d)] \leq \Lambda, \end{aligned}$$

as can be easily verified by distinguishing the cases $p_0 = 1$ and $p_0 \in (0, 1)$.

Similarly, in case of $p \in (0, 1)$, we necessarily have $p_0 \in (0, 1)$ and thus

$$\left| \frac{\sigma}{\tau} - \frac{d}{\min\{1, p\}} \right| = \left| \frac{d}{p} + |\mu| + \left\lceil |\mu| + \frac{d+\varepsilon}{p} \right\rceil - \frac{d}{p} \right| \leq \mu_0 + \left\lceil \mu_0 + \frac{d+\varepsilon}{p_0} \right\rceil,$$

which easily implies

$$|s^* - d\alpha_0(1 + \vartheta)| = \left| s^* - \frac{d\alpha_0}{\min\{1, p\}} \right| \leq \frac{1}{1 - \alpha} \left(s_0 + \frac{\alpha d}{p_0} \right) \leq \Lambda,$$

as well as

$$\begin{aligned} \left| s^* + \left(\frac{\sigma}{\tau} - d(1 + \vartheta) \right) \alpha_0 \right| &= \left| s^* + \alpha_0 \left(\frac{\sigma}{\tau} - \frac{d}{\min\{1, p\}} \right) \right| \\ &\leq \frac{1}{1 - \alpha} \left(s_0 + \alpha \left| \frac{\sigma}{\tau} - \frac{d}{\min\{1, p\}} \right| \right) \\ &\leq \frac{1}{1 - \alpha} \left[s_0 + \alpha \left(\mu_0 + \left\lceil \mu_0 + \frac{d+\varepsilon}{p_0} \right\rceil \right) \right] \leq \Lambda. \end{aligned}$$

All in all, our assumptions on M_0 thus yield

$$\begin{aligned} M_{00} + 1 &= d + 2 + \frac{d+1}{\min\{p_0, q_0\}} + \Lambda \\ &\geq d + 2 + \frac{d+1}{\tau} + \max \left\{ |s^* - d\alpha_0(1 + \vartheta)|, \left| s^* + \left(\frac{\sigma}{\tau} - d(1 + \vartheta) \right) \alpha_0 \right| \right\} \\ &= d + 2 + \frac{d+1}{\tau} + \max \left\{ |s^\natural + d\alpha_0|, \left| s^\natural + \left(d - \frac{\sigma}{\tau} \right) \alpha_0 \right| \right\}. \end{aligned}$$

Hence, Lemma 7.6 is applicable. For brevity, set $L_7 := 2^d \cdot (1 + (2 + 4r)^{\alpha_0} \cdot r)^{M_{00}+1+d}$. Using Lemma 7.6 and the estimate $|s^\sharp| \leq |s^*| + d\alpha_0 |\vartheta| \leq \frac{1}{1-\alpha} \left(s_0 + \alpha \frac{d}{p_0} \right)$ and setting $L_8 := \frac{\sigma_0}{\tau_0} + d \left(1 + \frac{1}{p_0} \right)$, we now get

$$\begin{aligned} K_1^{\frac{1}{\tau}} &\leq L_0 L_2 \cdot 4^{\frac{d\alpha_0}{p_0}} \cdot \left(\sup_{i \in \mathbb{Z}^d \setminus \{0\}} \sum_{j \in \mathbb{Z}^d \setminus \{0\}} M_{j,i}^{(0)} \right)^{1/\tau} \\ &\leq L_0 L_2 \cdot 4^{\frac{d\alpha_0}{p_0}} \cdot 2^{\frac{1}{\tau} + \frac{\sigma}{\tau} + |s^\sharp|} 6^{d/\tau} \cdot \max \left\{ 4^{\alpha_0 \left(\frac{\sigma}{\tau} + d \right) + |s^\sharp|} \cdot 12^{M_{00}+1} s_d, (2 + 4r)^{|s^\sharp| + \alpha_0 \left[\frac{\sigma}{\tau} + d + 1 + M_{00} \right]} L_7 \right\} \\ &\leq L_0 L_2 \cdot 6^{\frac{d}{\tau_0}} 2^{\frac{1}{\tau_0} + \frac{\sigma_0}{\tau_0} + \frac{1}{1-\alpha} \left(s_0 + 3\alpha \frac{d}{p_0} \right)} \cdot \max \left\{ 4^{\alpha_0 L_8 + \frac{s_0}{1-\alpha}} \cdot 12^{M_{00}+1} s_d, (2 + 4r)^{\frac{s_0}{1-\alpha} + \alpha_0 [L_8 + M_{00} + 1]} L_7 \right\}. \end{aligned}$$

Hence, $K_1^{1/\tau} \leq L_9$, for a constant L_9 which is independent of p, q, s, μ . Completely similar, we also get $K_2^{1/\tau} \leq L_9$, with the same constant. In particular, $K_1, K_2 < \infty$.

Having verified all prerequisites of Corollary 6.7, we get $\|\vec{C}\|^{\max\{1, \frac{1}{p}\}} \leq \Omega \cdot (K_1^{1/\tau} + K_2^{1/\tau}) \leq 2\Omega \cdot L_9$, where $\vec{C} : \ell_{[w(s^*)]^{\min\{1, p\}}}^r(\mathbb{Z}^d \setminus \{0\}) \rightarrow \ell_{[w(s^*)]^{\min\{1, p\}}}^r(\mathbb{Z}^d \setminus \{0\})$ is defined as in Assumption 5.1 (with $r := \max\{q, \frac{q}{p}\}$) and where

$$\begin{aligned} \Omega &= \Omega_0^K \Omega_1 \cdot (4 \cdot d)^{1+2 \lceil K + \frac{d+\varepsilon}{\min\{1, p\}} \rceil} \cdot \left(\frac{s_d}{\varepsilon} \right)^{1/\min\{1, p\}} \cdot \max_{|\beta| \leq \lceil K + \frac{d+\varepsilon}{\min\{1, p\}} \rceil} C^{(\beta)} \\ &\leq 2^{\mu_0} \cdot (4 \cdot d)^{1+2 \lceil \mu_0 + \frac{d+\varepsilon}{p_0} \rceil} \cdot \left(1 + \frac{s_d}{\varepsilon} \right)^{1/p_0} \cdot \max_{|\beta| \leq \lceil \mu_0 + \frac{d+\varepsilon}{p_0} \rceil} C^{(\beta)} =: L_{10}, \end{aligned}$$

where the constants $C^{(\beta)} = C^{(\beta)}(\Phi)$ are defined as in Assumption 6.1. Note again that L_{10} is independent of p, q, s, μ . Finally, Corollary 6.7 shows that the families $(\gamma_i)_{i \in \mathbb{Z}^d \setminus \{0\}}$, $(\gamma_{i,1})_{i \in \mathbb{Z}^d \setminus \{0\}}$ and $(\gamma_{i,2})_{i \in \mathbb{Z}^d \setminus \{0\}}$ fulfill Assumption 5.1 and that $(\gamma_i)_{i \in \mathbb{Z}^d \setminus \{0\}}$ satisfies Assumption 3.6, so that Theorem 5.6 is applicable.

This theorem shows that $\Gamma^{(\delta)}$ indeed forms an atomic decomposition for $M_{(s, \mu), \alpha}^{p, q}(\mathbb{R}^d) = \mathcal{D} \left(\mathcal{Q}_r^{(\alpha)}, L_{v(\mu)}^p, \ell_{w(s^*)}^q \right)$, as soon as $0 < \delta \leq \min\{1, \delta_{00}\}$, with

$$\delta_{00}^{-1} := \begin{cases} \frac{2s_d}{\sqrt{d}} \cdot (2^{17} \cdot d^2 \cdot (K+2+d))^{K+d+3} \cdot (1 + R_{\mathcal{Q}_r^{(\alpha)}})^{d+1} \cdot \Omega_0^{4K} \Omega_1^4 \Omega_2^{(p, K)} \Omega_4^{(p, K)} \cdot \|\vec{C}\|, & \text{if } p \geq 1, \\ \left(\frac{2^{14} \cdot d^{\frac{3}{2}}}{2^{45} \cdot d^{17}} \right)^{\frac{d}{p}} \cdot \left(\frac{s_d}{p} \right)^{\frac{1}{p}} \left(2^{68} \cdot d^{14} \cdot \left(K+1 + \frac{d+1}{p} \right)^3 \right)^{K+2 + \frac{d+1}{p}} \cdot (1 + R_{\mathcal{Q}_r^{(\alpha)}})^{1 + \frac{3d}{p}} \cdot \Omega_0^{16K} \Omega_1^{16} \Omega_2^{(p, K)} \Omega_4^{(p, K)} \cdot \|\vec{C}\|^{\frac{1}{p}}, & \text{if } p < 1. \end{cases}$$

Thus, to establish the claim of the current theorem, it suffices to show that δ_{00}^{-1} can be bounded independently of p, q, s, μ . Strictly speaking, we then still have to verify that the coefficient space used in Theorem 5.6 is just $\mathcal{C}_{p, q, s, \mu}^{(\alpha)}$, but this can be done precisely as in the proof of Theorem 7.7.

Now, for $p \geq 1$, we have because of $K = |\mu| \leq \mu_0$ and $\Omega_0 = 1$, as well as $\Omega_1 = 2^{\mu_0}$ and $\Omega_4^{(p, K)} \leq L_5$, as well as $\Omega_2^{(p, K)} \leq L_4$ that

$$\delta_{00}^{-1} \leq \frac{2s_d}{\sqrt{d}} \cdot (2^{17} \cdot d^2 \cdot (\mu_0 + 2 + d))^{\mu_0 + d + 3} \cdot (1 + R_{\mathcal{Q}_r^{(\alpha)}})^{d+1} \cdot L_4 L_5 \cdot 16^{\mu_0} \cdot 2L_9 L_{10},$$

which is independent of p, q, s, μ .

Finally, in case of $p \in (0, 1)$, we get similarly that

$$\delta_{00}^{-1} \leq \frac{(2^{14})^{\frac{d}{p_0}}}{2^{45} \cdot d^{17}} \cdot \left(1 + \frac{s_d}{p_0} \right)^{\frac{1}{p_0}} \left(2^{68} \cdot d^{14} \cdot \left(\mu_0 + 1 + \frac{d+1}{p_0} \right)^3 \right)^{\mu_0 + 2 + \frac{d+1}{p_0}} \cdot (1 + R_{\mathcal{Q}_r^{(\alpha)}})^{1 + \frac{3d}{p_0}} \cdot 2^{16\mu_0} \cdot L_4 L_5 \cdot 2L_9 L_{10},$$

which is independent of p, q, s, μ , as desired. \square

We close this section with an overview over the history and applications of α -modulation spaces and with a comparison of our results to the established literature.

Remark 7.9. α -modulation spaces were originally introduced in Gröbner's PhD. thesis [43], see also [24]. The definition of these spaces was motivated by the realization that both modulation spaces and (inhomogeneous) Besov spaces fit into the common framework of decomposition spaces, which Feichtinger and Gröbner developed at the time [24, 23]. The underlying frequency coverings are the uniform covering for the modulation spaces and

the dyadic covering for the Besov spaces. Given these two “end-point” types of spaces, the α -modulation spaces provide a continuous family of smoothness spaces which “lie between” modulation and Besov spaces in the sense that the associated α -modulation coverings $\mathcal{Q}_r^{(\alpha)}$ form a continuously indexed family of coverings which yields the uniform covering for $\alpha = 0$ and the (inhomogeneous) dyadic covering in the limit $\alpha \uparrow 1$. Note though that the β -modulation space $M_{s,\beta}^{p,q}(\mathbb{R}^d)$ *can not* be obtained by complex interpolation between two α -modulation spaces $M_{s_i,\alpha_i}^{p_i,q_i}(\mathbb{R}^d)$ with $\alpha_1 \neq \alpha_2$, except in a few trivial special cases[48]. We also remark that the frequency covering associated to the α -modulation spaces was independently introduced by Päiväranta and Somersalo[65, Lemma 2.1], in order to generalize the Calderón-Vaillancourt boundedness result for pseudodifferential operators to the local Hardy spaces h_p .

As for the classical modulation spaces, one application of α -modulation spaces is that they are suitable domains for the study of pseudodifferential operators: For the one-dimensional case, it was shown in [5] that if $T \in \text{OPS}_{\alpha,\delta}^{s_0}$, then $T : M_{s,\alpha}^{p,q}(\mathbb{R}) \rightarrow M_{s+\alpha-s_0-1,\alpha}^{p,q}(\mathbb{R})$ is continuous, if $p, q \in (1, \infty)$, $0 < \alpha \leq 1$, $0 \leq \delta \leq \alpha$ and $\delta < 1$. The multivariate case was considered in [7], where it was shown for symbols $\sigma \in S_{\varrho,0}^m$ and for $0 \leq \alpha \leq \varrho \leq 1$ that $\sigma(x, D) : M_{s,\alpha}^{p,q}(\mathbb{R}^d) \rightarrow M_{s-m,\alpha}^{p,q}(\mathbb{R}^d)$ is continuous for $p, q \in (1, \infty)$. For related results, see also [60].

The embeddings between α -modulation spaces and other function spaces have been considered by a number of authors: Already in Gröbner’s PhD. thesis [43], embeddings between $M_{s_1,\alpha_1}^{p,q}(\mathbb{R}^d)$ and $M_{s_2,\alpha_2}^{p,q}(\mathbb{R}^d)$ for $\alpha_1, \alpha_2 \in [0, 1]$ are considered, but the resulting criteria are not sharp. These non-sharp conditions were improved by Toft and Wahlberg[72], shortly before the question was completely solved by Han and Wang[50]. Note, however, that the preceding results only considered the embedding $M_{s_1,\alpha_1}^{p,q}(\mathbb{R}^d) \hookrightarrow M_{s_2,\alpha_2}^{p,q}(\mathbb{R}^d)$, where the exponents p, q are *the same* on both sides. The existence of the completely general embedding $M_{s_1,\alpha_1}^{p_1,q_1}(\mathbb{R}^d) \hookrightarrow M_{s_2,\alpha_2}^{p_2,q_2}(\mathbb{R}^d)$ was characterized completely in my PhD. thesis [76, Theorems 6.1.7 and 6.2.8]. The same results also appear in my recent paper [77, Theorems 9.7, 9.13, 9.14 and Corollary 9.16]. We finally remark that the complete characterization of the embedding $M_{s_1,\alpha_1}^{p_1,q_1}(\mathbb{R}^d) \hookrightarrow M_{s_2,\alpha_2}^{p_2,q_2}(\mathbb{R}^d)$ was also obtained independently in [49].

The existence of embeddings $M_{s_1,\alpha}^{p,q}(\mathbb{R}^d) \hookrightarrow L_{s_2}^p(\mathbb{R}^d)$ and $L_{s_1}^p(\mathbb{R}^d) \hookrightarrow M_{s_2,\alpha}^{p,q}(\mathbb{R}^d)$ between α -modulation spaces and Sobolev spaces (or Bessel potential spaces) has been fully characterized in [52]; in the same paper, the author also considers these embeddings when the Sobolev spaces $L_s^p(\mathbb{R}^d)$ are replaced by the local hardy spaces $h_p(\mathbb{R}^d)$, for $p \in (0, 1)$. Embeddings between α -modulation spaces $M_{s,\alpha}^{p_1,q}(\mathbb{R}^d)$ and the classical Sobolev spaces $W^{k,p_2}(\mathbb{R}^d)$ are also considered in [78, Example 7.3], as an application of a more general theory: For $p_2 \in [1, 2] \cup \{\infty\}$, a complete characterization of the existence of this embedding is obtained, but for $p_2 \in (2, \infty)$, the given sufficient criteria are strictly stronger than the given necessary criteria.

Finally, we discuss the existing results concerning Banach frames and atomic decompositions for α -modulation spaces. A large number of results in this direction were obtained by Borup and Nielsen: In [8], they showed that certain **brushlet orthonormal bases** of $L^2(\mathbb{R})$ form unconditional bases for the α -modulation spaces $M_{s,\alpha}^{p,q}(\mathbb{R})$, for $p, q \in (1, \infty)$. Furthermore, a characterization of the α -modulation (quasi)-norm in terms of the brushlet coefficients is obtained for arbitrary $p, q \in (0, \infty]$. In fact, Borup and Nielsen showed that brushlet bases even yield **greedy bases** (i.e., they are unconditional bases which satisfy the so-called **democracy condition**), which was then used to characterize the associated approximation spaces. As a further application of these brushlet bases for α -modulation spaces, Borup and Nielsen derived boundedness results for certain pseudodifferential operators, as briefly discussed above. In [61], Nielsen generalized the results concerning brushlet bases from the one-dimensional case to the case $d = 2$. Despite their great utility, we remark that brushlet bases are *not* generated by a single prototype function; furthermore, brushlets are bandlimited and can thus *not* be compactly supported.

In addition to these brushlet bases for α -modulation spaces in dimensions $d = 1$ and $d = 2$, Borup and Nielsen also proved existence of Banach frames for α -modulation spaces for the general case $d \in \mathbb{N}$. A first construction of a *non-tight* frame with explicitly given dual was obtained in [6] and then generalized in [9, Section 6.1] to obtain *tight* Banach frames, even for the case of general decomposition spaces, not just for α -modulation spaces. But again, the frames constructed in these two papers are *not* generated by a single prototype function and are bandlimited.

These limitations were partly overcome in later papers: In [63, Theorem 1.1], Nielsen and Rasmussen obtained compactly supported Banach frames for α -modulation spaces. These frames $(\psi_{k,n})_{k \in \mathbb{Z}^d \setminus \{0\}, n \in \mathbb{Z}^d}$, however, are *not* of the same structured form as in Theorems 7.7 and 7.8. Instead, $\psi_{k,n}(x) = e^{i\langle x, d_k \rangle} \sum_{\ell=1}^K a_{k,\ell} \cdot g(c_k x + b_{k,n,\ell})$ for suitable (but unknown) $K \in \mathbb{N}$, $a_{k,\ell} \in \mathbb{C}$, $d_k, b_{k,n,\ell} \in \mathbb{R}^d$ and $c_k \in \mathbb{R}^*$, which makes it hard to say anything about e.g. the time-frequency concentration of the family $(\psi_{k,n})_{k,n}$. This limitation was finally removed by Nielsen in [62], where it was shown—for a special class of coverings \mathcal{Q} , which includes the α coverings $\mathcal{Q}_r^{(\alpha)}$ for $0 \leq \alpha < 1$, cf. [62, Lemma 2.9]—that one can choose a single, compactly supported generator (or prototype) γ such that the resulting

family $\Gamma^{(\delta)}$ defined as in Theorem 7.7 yields a Banach frame for the α -modulation space $M_{s,\alpha}^{p,q}(\mathbb{R}^d)$. The main difference in comparison to Theorems 7.7 and 7.8 is that [62] merely establishes *existence* of a suitable generator γ ; it does *not* provide readily verifiable conditions on γ which allow to decide if γ is suitable. This is due to the employed proof technique: Nielsen first shows that a certain bandlimited generator γ_0 generates a Banach frame and then uses a perturbation argument to show that γ generates a Banach frame, provided that γ is close enough to γ_0 in a certain sense. The most concrete criterion in [62] concerning γ is that under certain *readily verifiable* conditions on a “prototypical prototype” g (cf. [62, equations (3.13), (3.14)]), one can always obtain a *suitable* γ by taking linear combinations of translations of g . In stark contrast, Theorems 7.7 and 7.8 (plus the associated remark) show that *any* compactly supported prototype γ generates Banach frames and atomic decompositions for the α -modulation spaces, assuming that γ is sufficiently smooth and has nonvanishing Fourier transform on a certain neighborhood of the origin.

In addition to the constructions by Borup, Nielsen and Rasmussen, Banach frames for α -modulation spaces have also been considered by Fornasier[30], by Dahlke et al.[11] and finally by Speckbacher et al.[71], all for the case $d = 1$ and $p, q \in [1, \infty]$. The idea in [30] is to show that a family $\Gamma_\alpha^{(\delta)}$ similar to $\Gamma^{(\delta)}$ from Theorem 7.7 is **intrinsically self-localized**, under suitable readily verifiable assumptions on the generator γ , so that, for a sufficiently small sampling density, the family $\Gamma_\alpha^{(\delta)}$ forms a Banach frame and an atomic decomposition for the α -modulation space $M_{s,\alpha}^{p,q}(\mathbb{R})$. In particular, γ can be taken to have compact support, since any Schwartz function with $\hat{\gamma}(\xi) \neq 0$ on $[-1, 1]$ is suitable, cf. [30, Theorem 3.4]. Hence, Fornasier’s results are very similar to Theorems 7.7 and 7.8; the main difference is that the results in this paper apply for the full range $p, q \in (0, \infty]$ and also for $d > 1$. In this context, we remark that Fornasier notes that “We expect that the approach illustrated in this paper [i.e., in [30]] can be useful also for a frame characterization of $M_{p,q}^{s,\alpha}(\mathbb{R}^d)$ for $d > 1$, with major technical difficulties.” A further difference is that Fornasier only requires decay of $\hat{\gamma}$, not of $\partial^\alpha \hat{\gamma}$, cf. [30, equation (29)].

The approach by Dahlke et al.[11] is still different: They consider **coorbit spaces** of certain quotients of the **affine Weyl-Heisenberg group**, based on the general theory of coorbit spaces of homogeneous spaces developed in [15, 16]. It is shown in [11] that the general theory applies in this setting and (for suitable quotients) that the associated coorbit spaces coincide with certain α -modulation spaces. Precisely, [11, Theorem 6.1] shows that $\mathcal{H}_{p,v_{s-\alpha(1/p-1/2)},\alpha} = M_{s,\alpha}^{p,p}(\mathbb{R})$, up to trivial identifications. Note that this only includes α -modulation spaces with $p = q$. The same theorem also shows—as a consequence of the general discretization theory for coorbit spaces—that one obtains Banach frames and atomic decompositions for the spaces $\mathcal{H}_{p,v_{s-\alpha(1/p-1/2)},\alpha} = M_{s,\alpha}^{p,p}(\mathbb{R})$ which are of a similar form as the family $\Gamma^{(\delta)}$ considered in Theorems 7.7 and 7.8. The main difference of these two theorems to the results in [11] is that [11] is only applicable to *bandlimited generators*, only for $p = q \in [1, \infty]$ and only for $d = 1$. Furthermore, while in Theorems 7.7 and 7.8 only the sampling density *in the space domain* has to be sufficiently dense, in [11] one needs to adjust the sampling density *in space and in frequency*. Note though that the assumption $\hat{\gamma}(\xi) \neq 0$ on $\overline{B_r}(0)$ is not present in [11].

Finally, in [71], Speckbacher et al. extended the results of [11] by showing that the theory developed in [15, 16, 11] is also applicable for suitable *compactly supported* generators, only subject to certain decay and smoothness conditions, cf. [71, Theorem 5.9]. Note though that this does not remove the assumptions $p = q \in [1, \infty]$ and $d = 1$. Precisely, the condition on the generator $\gamma \in L^2(\mathbb{R})$ imposed in [71] to generate a Banach frame and an atomic decomposition for $\mathcal{H}_{p,v_{s-\alpha(1/p-1/2)},\alpha} = M_{s,\alpha}^{p,p}(\mathbb{R})$ is that $\hat{\gamma} \in C^3(\mathbb{R})$ with $|\partial^j \hat{\gamma}(\xi)| \lesssim (1 + |\xi|)^{-r}$ for $j \in \{0, 1, 2, 3\}$ and

$$r > 1 + \frac{2 + 2[s - \alpha(1/p - 1/2)] + 7\alpha - 4\alpha^2}{2(1 - \alpha)^2} = 1 + \frac{1 + s + \alpha\left(4 - \frac{1}{p}\right) - 2\alpha^2}{(1 - \alpha)^2} \geq \frac{2 + s + \alpha(1 - \alpha)}{(1 - \alpha)^2}.$$

In comparison, at least for $\gamma \in C_c^1(\mathbb{R})$, Theorem 7.7 (with $\mu_0 = 0$, $p_0 = q_0 = 1$, $s_0 = s_1 = s$ and $\varepsilon = \frac{1}{2}$) only requires $|\partial^j \hat{\gamma}(\xi)| \lesssim (1 + |\xi|)^{-N_0}$ for $j \in \{0, 1, 2\}$ and

$$N_0 = 5 + \frac{1}{1 - \alpha} \cdot \max\{s + \alpha, s + 2\alpha\} = 5 + \frac{s + 2\alpha}{1 - \alpha} = \frac{5 + s - 3\alpha}{1 - \alpha}$$

and Theorem 7.8 requires the same, but with

$$N_0 = 2 \cdot (3 + \varepsilon) + \frac{s + \alpha}{1 - \alpha} = 6 + 2\varepsilon + \frac{s + \alpha}{1 - \alpha},$$

where $\varepsilon \in (0, 1)$ can be chosen arbitrarily small. Hence, in particular for $\alpha \approx 1$ or for $\alpha > 0$ and large s , the prerequisites of Theorems 7.7 and 7.8 are easier to fulfill than those in [71]. We finally remark that Fornasier[30] requires $|\hat{\gamma}(\xi)| \lesssim (1 + |\xi|)^{-M_0}$ where M_0 satisfies $M_0 \geq 5 + \frac{1}{2} + \frac{2s+2\alpha}{1-\alpha}$, which is very close to the conditions in Theorems 7.7 and 7.8.

Hence, our very general theory yields results comparable to specialized treatments like [30] and (in many cases) better results than those obtained by using general coorbit theory. Further, it is applicable for general $d \in \mathbb{N}$ and also for general $p, q \in (0, \infty]$ instead of only for $p = q \in [1, \infty]$. \blacklozenge

8. EXISTENCE OF COMPACTLY SUPPORTED BANACH FRAMES AND ATOMIC DECOMPOSITIONS FOR INHOMOGENEOUS BESOV SPACES

In this section, we investigate assumptions on the **scaling function** $\varphi : \mathbb{R}^d \rightarrow \mathbb{C}$ and the **mother wavelet** $\psi : \mathbb{R}^d \rightarrow \mathbb{C}$ which ensure that the associated inhomogeneous **wavelet system** with sampling density $c > 0$,

$$W(\varphi, \psi; c) := (\varphi(\bullet - c \cdot k))_{k \in \mathbb{Z}^d} \cup \left(2^{j \frac{d}{2}} \cdot \psi(2^j \bullet - c \cdot k) \right)_{j \in \mathbb{N}, k \in \mathbb{Z}^d},$$

generates Banach frames or atomic decompositions for a subclass of the class of inhomogeneous **Besov spaces**.

The inhomogeneous Besov spaces are decomposition spaces which are defined using a certain dyadic covering of \mathbb{R}^d , which we introduce now.

Lemma 8.1. *For $j \in \mathbb{N}$, let $T_j := 2^j \cdot \text{id}$ and $b_j := 0$, as well as $Q'_j := B_4(0) \setminus \overline{B_{1/4}(0)}$. Furthermore, set $T_0 := \text{id}$ and $b_0 := 0$, as well as $Q'_0 := B_2(0)$. The **(inhomogeneous) Besov covering** of \mathbb{R}^d is given by*

$$\mathcal{B} := (Q_j)_{j \in \mathbb{N}_0} := (T_j Q'_j + b_j)_{j \in \mathbb{N}_0}.$$

This covering is a semi-structured admissible covering of \mathbb{R}^d . Furthermore, \mathcal{B} admits a regular partition of unity $\Phi = (\varphi_j)_{j \in \mathbb{N}_0}$ (which thus fulfills Assumption 6.1), which we fix for the remainder of the section.

Finally, \mathcal{B} fulfills the standing assumptions from Section 1.3; in particular, $\|T_j^{-1}\| \leq 1 =: \Omega_0$ for all $j \in \mathbb{N}_0$. \blacktriangleleft

Proof. It was shown in [78, Example 7.2] that \mathcal{B} is a semi-structured covering of \mathbb{R}^d . In the same example, it was also shown that \mathcal{B} is in fact a regular covering of \mathbb{R}^d , i.e., \mathcal{B} admits a regular partition of unity Φ , as claimed. Thanks to Corollary 6.5, Φ is also a \mathcal{B} - v_0 -BAPU for each weight v_0 satisfying the assumptions from Section 1.3.

To verify the standing assumptions from Section 1.3 pertaining to the covering $\mathcal{Q} = \mathcal{B}$, we thus only have to verify $\|T_j^{-1}\| \leq 1$ for all $j \in \mathbb{N}_0$. But since $T_j = 2^j \cdot \text{id}$ for all $j \in \mathbb{N}_0$, we simply have $\|T_j^{-1}\| = 2^{-j} \leq 1$, as claimed. \square

Now, we can define the inhomogeneous Besov spaces:

Definition 8.2. For $p, q \in (0, \infty]$ and $s, \mu \in \mathbb{R}$, we define the associated **(weighted) inhomogeneous Besov space** as

$$\mathcal{B}_{s, \mu}^{p, q}(\mathbb{R}^d) := \mathcal{D} \left(\mathcal{B}, L_{v(\mu)}^p, \ell_{(2^{js})}^q \right)_{j \in \mathbb{N}_0},$$

with $v(\mu)$ as in Lemma 7.4. The **classical inhomogeneous Besov spaces** are given by $\mathcal{B}_s^{p, q}(\mathbb{R}^d) := \mathcal{B}_{s, 0}^{p, q}(\mathbb{R}^d)$. \blacktriangleleft

Remark. • It is not hard to see that the weight $(2^{js})_{j \in \mathbb{N}_0}$ is indeed \mathcal{B} -moderate (cf. equation (1.13)). Furthermore, we saw in Lemma 7.4 that the weight $v(\mu)$ satisfies all assumptions from Section 1.3, with $K := |\mu|$ and $\Omega_1 := 2^{|\mu|}$, as well as $v_0 : \mathbb{R}^d \rightarrow (0, \infty)$, $x \mapsto [2 \cdot (1 + |x|)]^{|\mu|}$. All in all, we thus see that all standing assumptions from Section 1.3 are satisfied. In particular, Lemma 5.5 and Proposition 2.24 show that the spaces $\mathcal{B}_{s, \mu}^{p, q}(\mathbb{R}^d)$ are well-defined Quasi-Banach spaces.

• It is not hard to see that the usual inhomogeneous Besov spaces (e.g. as defined in [42, Definition 6.5.1]) coincide with the spaces $\mathcal{B}_s^{p, q}(\mathbb{R}^d)$ defined above, up to trivial identifications. The main difference is that the usual Besov spaces are defined as subspaces of the space of tempered distributions, while $\mathcal{B}_s^{p, q}(\mathbb{R}^d)$ is a subspace of $Z'(\mathbb{R}^d) = [\mathcal{F}(C_c^\infty(\mathbb{R}^d))]'$, cf. Subsection 1.3. Claiming that the two spaces coincide amounts to claiming that each $f \in \mathcal{B}_s^{p, q}(\mathbb{R}^d)$ extends to a (uniquely determined) tempered distribution. As shown in [77, Lemma 9.15], this is indeed fulfilled. \blacklozenge

Note that if we define the family $\Gamma = (\gamma_j)_{j \in \mathbb{N}_0}$ by $\gamma_0 := \varphi$ and $\gamma_j := \psi$ for $j \in \mathbb{N}$, where $\varphi, \psi : \mathbb{R}^d \rightarrow \mathbb{C}$ are given, then the family $\Gamma^{(\delta)} = \left(L_{\delta \cdot T_j^{-T} k} \gamma^{[j]} \right)_{j \in \mathbb{N}_0, k \in \mathbb{Z}^d}$ considered in Theorem 5.6 (and in a slightly modified form also in Theorem 4.7) satisfies

$$\Gamma^{(\delta)} = (\varphi(\bullet - \delta k))_{k \in \mathbb{Z}^d} \cup \left(2^{j \frac{d}{2}} \cdot \psi(2^j \bullet - \delta k) \right) = W(\varphi, \psi; \delta),$$

at least up to an obvious re-indexing. Consequently, we can use Corollaries 6.6 and 6.7 to derive conditions on φ, ψ which ensure that the wavelet system $W(\varphi, \psi; \delta)$ yields a Banach frame, or an atomic decomposition for the (weighted) inhomogeneous Besov spaces $\mathcal{B}_{s, \mu}^{p, q}(\mathbb{R}^d)$. We begin with the case of Banach frames.

Proposition 8.3. *Let $p_0, q_0 \in (0, 1]$, $\varepsilon > 0$, $\mu_0 \geq 0$ and $-\infty < s_0 \leq s_1 < \infty$. Assume that $\varphi, \psi : \mathbb{R}^d \rightarrow \mathbb{C}$ satisfy the following conditions:*

- (1) *We have $\varphi, \psi \in L^1_{(1+|\bullet|)^{\mu_0}}(\mathbb{R}^d)$ and $\widehat{\varphi}, \widehat{\psi} \in C^\infty(\mathbb{R}^d)$, where all partial derivatives of $\widehat{\varphi}$ and $\widehat{\psi}$ are polynomially bounded.*
- (2) *We have $\varphi, \psi \in C^1(\mathbb{R}^d)$ and $\nabla \varphi, \nabla \psi \in L^1_{(1+|\bullet|)^{\mu_0}}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.*
- (3) *We have $\widehat{\varphi}(\xi) \neq 0$ for all $\xi \in \overline{B_2(0)}$ and $\widehat{\psi}(\xi) \neq 0$ for all $\xi \in \overline{B_4(0)} \setminus B_{1/4}(0)$.*
- (4) *We have*

$$\begin{aligned} |\partial^\alpha \widehat{\varphi}(\xi)| &\leq G_1 \cdot (1 + |\xi|)^{-L}, \\ |\partial^\alpha \widehat{\psi}(\xi)| &\leq G_2 \cdot (1 + |\xi|)^{-L_1} \cdot \min \left\{ 1, |\xi|^{L_2} \right\} \end{aligned}$$

for all $\xi \in \mathbb{R}^d$ and all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq \left\lceil \mu_0 + \frac{d+\varepsilon}{p_0} \right\rceil$ for suitable $G_1, G_2 > 0$ and certain $L_2 \geq 0$ and $L, L_1 \geq 1$ which satisfy

$$L > 1 - s_0 + \vartheta, \quad L_1 > 1 - s_0 + \vartheta, \quad \text{and} \quad L_2 > s_1,$$

$$\text{where } \vartheta := \frac{d}{p_0} + \mu_0 + \left\lceil \mu_0 + \frac{d+\varepsilon}{p_0} \right\rceil.$$

Then there is some $\delta_0 = \delta_0(p_0, q_0, s_0, s_1, \mu_0, \varepsilon, d, \varphi, \psi) > 0$ such that for each $0 < \delta \leq \delta_0$, the family

$$\Gamma^{(\delta)} = \left(2^{j\frac{d}{2}} \cdot \widetilde{\gamma}_j(2^j \bullet - \delta k) \right)_{j \in \mathbb{N}_0, k \in \mathbb{Z}^d}, \quad \text{with} \quad \widetilde{\gamma}_0 := \varphi(-\bullet) \text{ and } \widetilde{\gamma}_j := \psi(-\bullet) \text{ for } j \in \mathbb{N},$$

forms a Banach frame for $\mathcal{B}_{s,\mu}^{p,q}(\mathbb{R}^d)$, for arbitrary $p, q \in (0, \infty]$ and $s, \mu \in \mathbb{R}$ satisfying $p \geq p_0$, $q \geq q_0$, $s_0 \leq s \leq s_1$ and $|\mu| \leq \mu_0$.

Precisely, this means the following: Define the coefficient space

$$\mathcal{C}_{p,q,s,\mu} := \ell^q \left(2^{j(s+d(\frac{1}{2}-\frac{1}{p}))} \right)_{j \in \mathbb{N}_0} \left(\left[\ell^p_{[(1+|k|/2^j)^\mu]}(\mathbb{Z}^d) \right]_{k \in \mathbb{Z}^d} \right)_{j \in \mathbb{N}_0} \leq \mathbb{C}^{\mathbb{N}_0 \times \mathbb{Z}^d}.$$

Then the following hold:

- **The analysis operator**

$$A^{(\delta)} : \mathcal{B}_{s,\mu}^{p,q}(\mathbb{R}^d) \rightarrow \mathcal{C}_{p,q,s,\mu}, f \mapsto \left[\left(\left[2^{j\frac{d}{2}} \cdot \gamma_j(2^j \bullet) \right] * f \right) \left(\delta \cdot \frac{k}{2^j} \right) \right]_{j \in \mathbb{N}_0, k \in \mathbb{Z}^d}$$

is well-defined and bounded for all $0 < \delta \leq 1$. Here, $\gamma_0 := \varphi$ and $\gamma_j := \psi$ for $j \in \mathbb{N}$. The convolution considered here is defined as in equation (4.8).

- For $0 < \delta \leq 1$, there is a bounded linear **reconstruction operator** $R^{(\delta)} : \mathcal{C}_{p,q,s,\mu} \rightarrow \mathcal{B}_{s,\mu}^{p,q}(\mathbb{R}^d)$ satisfying $R^{(\delta)} \circ A^{(\delta)} = \text{id}_{\mathcal{B}_{s,\mu}^{p,q}}$. Furthermore, the action of $R^{(\delta)}$ on a given sequence is independent of the precise choice of p, q, s, μ .
- We have the following **consistency statement**: If $f \in \mathcal{B}_{s,\mu}^{p,q}(\mathbb{R}^d)$ and if $p_0 \leq \tilde{p} \leq \infty$ and $q_0 \leq \tilde{q} \leq \infty$ and if furthermore $s_0 \leq \tilde{s} \leq s_1$ and $|\tilde{\mu}| \leq \mu_0$, then the following equivalence holds:

$$f \in \mathcal{B}_{\tilde{s},\tilde{\mu}}^{\tilde{p},\tilde{q}}(\mathbb{R}^d) \quad \Longleftrightarrow \quad A^{(\delta)} f \in \mathcal{C}_{\tilde{p},\tilde{q},\tilde{s},\tilde{\mu}}. \quad \blacktriangleleft$$

Proof. Let p, q, s, μ as in the statement of the proposition. Our first goal is to provide suitable estimates for the quantity $M_{j,i}$ which appears in Corollary 6.6, i.e.,

$$M_{j,i} = \left(\frac{w_j}{w_i} \right)^\tau \cdot (1 + \|T_j^{-1} T_i\|)^\sigma \cdot \max_{|\beta| \leq 1} \left(|\det T_i|^{-1} \cdot \int_{Q_i} \max_{|\alpha| \leq N} \left| (\partial^\alpha \widehat{\partial^\beta \gamma_j})(S_j^{-1} \xi) \right| d\xi \right)^\tau,$$

where $K = |\mu|$ (cf. the remark after Definition 8.2) and

$$\begin{aligned} N &= \left\lceil K + \frac{d+\varepsilon}{\min\{1, p\}} \right\rceil, \\ \tau &= \min\{1, p, q\}, \\ \sigma &= \tau \cdot \left(\frac{d}{\min\{1, p\}} + K + \left\lceil K + \frac{d+\varepsilon}{\min\{1, p\}} \right\rceil \right). \end{aligned}$$

We immediately observe $N \leq N_0 := \left\lceil \mu_0 + \frac{d+\varepsilon}{p_0} \right\rceil$, so that our estimates regarding $|\partial^\alpha \widehat{\varphi}(\xi)|$ and $|\partial^\alpha \widehat{\psi}(\xi)|$ can be applied.

Indeed, since $\varphi, \psi \in C^1(\mathbb{R}^d)$ with $\nabla\varphi, \nabla\psi \in L^1_{(1+|\cdot|)^{\mu_0}}(\mathbb{R}^d) \hookrightarrow L^1(\mathbb{R}^d)$ and since $\gamma_j = \varphi$ for $j = 0$ and $\gamma_j = \psi$ otherwise, standard properties of the Fourier transform show

$$\widehat{\partial^\beta \gamma_j}(\xi) = (2\pi i \xi)^\beta \cdot \widehat{\gamma_j}(\xi) \quad \forall \xi \in \mathbb{R}^d \quad \forall \beta \in \mathbb{N}_0^d \text{ with } |\beta| \leq 1.$$

Consequently, we get for $i \in \mathbb{N}$ that

$$M_{j,i} \leq 2^{\tau s(j-i)} \cdot (1 + 2^{i-j})^\sigma \cdot \left(2\pi \cdot \max_{|\beta| \leq 1} 2^{-i \cdot d} \int_{2^{i-2} < |\eta| < 2^{i+2}} \max_{|\alpha| \leq N} |(\partial^\alpha [\xi \mapsto \xi^\beta \cdot \widehat{\gamma_j}(\xi)]) (\eta/2^j)| \, d\eta \right)^\tau. \quad (8.1)$$

Next, we observe for $\beta \in \mathbb{N}_0^d$ with $|\beta| = 1$, i.e., $\beta = e_j$ for some $j \in \underline{d}$, that

$$|\partial^\nu \xi^\beta| = \begin{cases} |\xi^\beta| \leq |\xi| \leq 1 + |\xi|, & \text{if } \nu = 0, \\ 1 \leq 1 + |\xi|, & \text{if } \nu = e_j, \\ 0 \leq 1 + |\xi| & \text{otherwise.} \end{cases}$$

Likewise, in case of $\beta = 0$, we have

$$|\partial^\nu \xi^\beta| = \begin{cases} 1 \leq 1 + |\xi|, & \text{if } \nu = 0, \\ 0 \leq 1 + |\xi|, & \text{otherwise.} \end{cases}$$

In connection with Leibniz's rule and the d -dimensional binomial theorem (cf. [29, Section 8.1, Exercise 2.b]), this yields for $j = 0$ and $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq N \leq N_0$ that

$$\begin{aligned} |\partial^\alpha [\xi \mapsto \xi^\beta \cdot \widehat{\gamma_0}(\xi)](\eta)| &\leq \sum_{\nu \leq \alpha} \binom{\alpha}{\nu} \cdot |\partial^\nu \eta^\beta| \cdot |\partial^{\alpha-\nu} \widehat{\gamma_0}(\eta)| \\ &\stackrel{(\text{assumption for } \widehat{\gamma_0}=\widehat{\varphi})}{\leq} G_1 \cdot (1 + |\eta|)^{1-L} \cdot \sum_{\nu \leq \alpha} \binom{\alpha}{\nu} \\ &\leq 2^{N_0} G_1 \cdot (1 + |\eta|)^{1-L}. \end{aligned} \quad (8.2)$$

Likewise, for $j \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq N \leq N_0$, we get

$$\begin{aligned} |\partial^\alpha [\xi \mapsto \xi^\beta \cdot \widehat{\gamma_j}(\xi)](\eta)| &\leq \sum_{\nu \leq \alpha} \binom{\alpha}{\nu} \cdot |\partial^\nu \eta^\beta| \cdot |\partial^{\alpha-\nu} \widehat{\gamma_j}(\eta)| \\ &\stackrel{(\text{assumption for } \widehat{\gamma_j}=\widehat{\psi})}{\leq} G_2 \cdot (1 + |\eta|)^{1-L_1} \cdot \min\{1, |\eta|^{L_2}\} \cdot \sum_{\nu \leq \alpha} \binom{\alpha}{\nu} \\ &\leq 2^{N_0} G_2 \cdot (1 + |\eta|)^{1-L_1} \cdot \min\{1, |\eta|^{L_2}\}. \end{aligned} \quad (8.3)$$

Now, we first consider the case $i \in \mathbb{N}$ and note

$$\begin{aligned} \lambda_d(\{\eta \in \mathbb{R}^d \mid 2^{i-2} < |\eta| < 2^{i+2}\}) &= v_d \cdot (2^{d(i+2)} - 2^{d(i-2)}) \\ &= 2^{i \cdot d} \cdot v_d (4^d - 4^{-d}) \\ &\leq 2^{i \cdot d} \cdot 4^d v_d, \end{aligned} \quad (8.4)$$

so that

$$2^{-i \cdot d} \cdot \int_{2^{i-2} < |\eta| < 2^{i+2}} h(\eta) \, d\eta \leq 4^d v_d \cdot \sup_{2^{i-2} < |\eta| < 2^{i+2}} h(\eta) \quad (8.5)$$

for each nonnegative (measurable) function h . Now, we distinguish two subcases for estimating $M_{j,i}$.

Case 1: We have $j \in \mathbb{N}$. In this case, we distinguish two additional subcases:

- (1) We have $j \leq i$. For $2^{i-2} < |\eta| < 2^{i+2}$, this implies $|\eta/2^j| \geq 2^{i-2-j} = \frac{2^{i-j}}{4} = \frac{2^{|i-j|}}{4}$. Since we have $L_1 \geq 1$, a combination of this estimate with equations (8.1), (8.3) and (8.5) yields

$$\begin{aligned} M_{j,i} &\leq 2^\sigma \cdot 2^{-\tau s|j-i|} \cdot 2^{\sigma|i-j|} \cdot \left(2^{N_0} G_2 \cdot 2\pi \cdot 4^d v_d \cdot 4^{L_1-1} \cdot 2^{|i-j|(1-L_1)} \right)^\tau \\ &\leq 2^\sigma \cdot (2^{N_0} G_2 \cdot 2\pi \cdot 4^d v_d \cdot 4^{L_1})^\tau \cdot 2^{|i-j|[\sigma-\tau(L_1-1+s)]} \\ &=: 2^\sigma \cdot H_1^\tau \cdot 2^{|i-j|[\sigma-\tau(L_1-1+s)]}. \end{aligned}$$

(2) We have $i \leq j$. For $2^{i-2} < |\eta| < 2^{i+2}$, this implies because of $L_2 \geq 0$ that

$$\min \left\{ 1, |\eta/2^j|^{L_2} \right\} \leq (2^{i+2-j})^{L_2} \leq 4^{L_2} \cdot 2^{-L_2|i-j|}.$$

Furthermore, $(1 + |\eta/2^j|)^{1-L_1} \leq 1$, since $L_1 \geq 1$. Hence, as in the previous case, we can combine equations (8.1), (8.3) and (8.5) to derive

$$\begin{aligned} M_{j,i} &\leq 2^{\tau s|i-j|} \cdot 2^\sigma \cdot \left(2^{N_0} G_2 \cdot 2\pi \cdot 4^d v_d \cdot 4^{L_2} \cdot 2^{-L_2|i-j|} \right)^\tau \\ &=: 2^\sigma \cdot H_2^\tau \cdot 2^{\tau|i-j|(s-L_2)}. \end{aligned}$$

Case 2: We have $j = 0$. In this case, we have for $2^{i-2} < |\eta| < 2^{i+2}$ that $|\eta/2^j| = |\eta| \geq 2^{i-2}$ and hence $(1 + |\eta/2^j|)^{1-L} \leq 4^{L-1} 2^{-i(L-1)} \leq 4^L \cdot 2^{-(L-1)|i-j|}$. Here, we used $L \geq 1$.

Now, a combination of equations (8.1), (8.2) and (8.5) yields

$$\begin{aligned} M_{0,i} &\leq 2^{-\tau s|i-j|} \cdot 2^\sigma 2^{\sigma|i-j|} \cdot \left(2^{N_0} G_1 \cdot 2\pi \cdot 4^d v_d \cdot 4^L \cdot 2^{-(L-1)|i-j|} \right)^\tau \\ &=: 2^\sigma \cdot H_3^\tau \cdot 2^{|i-j|[\sigma-\tau(L-1+s)]}. \end{aligned}$$

These are the desired estimates in case of $i \in \mathbb{N}$. It remains to consider the case $i = 0$. Here, equation (8.1) takes on the slightly modified form

$$\begin{aligned} M_{j,0} &\leq 2^{\tau s j} \cdot (1 + 2^{-j})^\sigma \cdot \left(2\pi \cdot \max_{|\beta| \leq 1} \int_{B_2(0)} \max_{|\alpha| \leq N} |(\partial^\alpha [\xi \mapsto \xi^\beta \cdot \widehat{\gamma}_j(\xi)]) (\eta/2^j)| \, d\eta \right)^\tau \\ &\leq 2^\sigma \cdot 2^{\tau s j} \cdot \left(2\pi \cdot \lambda_d(B_2(0)) \cdot \max_{|\beta| \leq 1} \sup_{|\eta| < 2} |(\partial^\alpha [\xi \mapsto \xi^\beta \cdot \widehat{\gamma}_j(\xi)]) (\eta/2^j)| \right)^\tau. \end{aligned} \quad (8.6)$$

Now, we again distinguish two cases:

Case 1: We have $j \in \mathbb{N}$. Here, we observe for $|\eta| < 2$ that

$$\min \left\{ 1, |\eta/2^j|^{L_2} \right\} \leq 2^{(1-j)L_2} = 2^{L_2} \cdot 2^{-L_2|i-j|}.$$

Since we also have $L_1 \geq 1$, a combination of equations (8.6) and (8.3) yields

$$\begin{aligned} M_{j,0} &\leq 2^\sigma \cdot 2^{\tau s|i-j|} \cdot \left(2^{N_0} G_2 \cdot 2\pi \cdot \lambda_d(B_2(0)) \cdot 2^{L_2} \cdot 2^{-L_2|i-j|} \right)^\tau \\ &=: 2^\sigma \cdot H_4^\tau \cdot 2^{\tau|i-j|(s-L_2)}. \end{aligned}$$

Case 2: We have $j = 0$. Because of $L \geq 1$, we have $(1 + |\eta/2^j|)^{1-L} \leq 1$ for arbitrary $\eta \in \mathbb{R}^d$, so that a combination of equations (8.6) and (8.2) yields

$$\begin{aligned} M_{0,0} &\leq 2^\sigma \cdot (2^{N_0} G_1 \cdot 2\pi \cdot \lambda_d(B_2(0)))^\tau \\ &=: 2^\sigma \cdot H_5^\tau \\ &= 2^\sigma \cdot H_5^\tau \cdot 2^{-|i-j|\zeta}, \end{aligned}$$

where $\zeta \in \mathbb{R}$ can be chosen arbitrarily, since $|i-j| = 0$.

All in all, if we set $H_6 := \max \{H_1, \dots, H_5\}$, a combination of the preceding cases shows

$$M_{j,i} \leq 2^\sigma \cdot H_6^\tau \cdot 2^{-\tau|i-j| \min\{1, L_2-s, L-1+s-\frac{\sigma}{\tau}, L_1-1+s-\frac{\sigma}{\tau}\}}. \quad (8.7)$$

Note that H_1, \dots, H_5 , and hence also H_6 , are all independent of p, q, μ, s .

Furthermore, we have

$$\frac{\sigma}{\tau} = \frac{d}{\min\{1, p\}} + K + \left\lceil K + \frac{d+\varepsilon}{\min\{1, p\}} \right\rceil \leq \frac{d}{p_0} + \mu_0 + \left\lceil \mu_0 + \frac{d+\varepsilon}{p_0} \right\rceil = \vartheta$$

and $\sigma = \tau \cdot \frac{\sigma}{\tau} \leq \tau \vartheta$, as well as $s_0 \leq s \leq s_1$. Hence,

$$M_{j,i} \leq (2^\vartheta \cdot H_6)^\tau \cdot 2^{-\tau|i-j| \min\{1, L_2-s_1, L-1+s_0-\vartheta, L_1-1+s_0-\vartheta\}}.$$

But our assumptions on L, L_1, L_2 imply that the exponent $\lambda := \min\{1, L_2 - s_1, L - 1 + s_0 - \vartheta, L_1 - 1 + s_0 - \vartheta\}$ is positive. Hence, we get, for the constants C_1, C_2 defined in Corollary 6.6,

$$\begin{aligned}
C_1^{1/\tau} &= \left(\sup_{i \in \mathbb{N}_0} \sum_{j \in \mathbb{N}_0} M_{j,i} \right)^{1/\tau} \leq 2^\vartheta \cdot H_6 \cdot \sup_{i \in \mathbb{N}_0} \left(\sum_{j \in \mathbb{N}_0} 2^{-\tau\lambda|i-j|} \right)^{1/\tau} \\
&\quad (\text{for } \ell=i-j) \leq 2^\vartheta \cdot H_6 \cdot \left(\sum_{\ell \in \mathbb{Z}} 2^{-\tau\lambda|\ell|} \right)^{1/\tau} \\
&\leq 2^\vartheta \cdot H_6 \cdot \left(2 \cdot \sum_{\ell=0}^{\infty} 2^{-\tau\lambda\ell} \right)^{1/\tau} \\
&\quad (\text{since } \ell^{\tau_0} \hookrightarrow \ell^\tau \text{ is norm-decreasing for } \tau_0 := \min\{p_0, q_0\}) \leq 2^\vartheta \cdot H_6 \cdot 2^{1/\tau} \cdot \left(\sum_{\ell=0}^{\infty} 2^{-\tau_0\lambda\ell} \right)^{1/\tau_0} \\
&\leq 2^{\vartheta + \frac{1}{\tau_0}} \cdot H_6 \cdot \left(\frac{1}{1 - 2^{-\tau_0\lambda}} \right)^{1/\tau_0} =: H_7 < \infty.
\end{aligned}$$

Observe again that H_7 is independent of p, q, μ, s . Exactly the same estimate also yields $C_2^{1/\tau} \leq H_7$.

Next, we set $\gamma_1^{(0)} := \varphi$ and $\gamma_2^{(0)} := \psi$. Furthermore, we set $n_j := 2$ for $j \in \mathbb{N}$ and $n_0 := 1$, so that $\gamma_j = \gamma_{n_j}^{(0)}$ for all $j \in \mathbb{N}_0$. In the notation of Lemma 3.7, these definitions entail

$$\begin{aligned}
Q^{(1)} &= \bigcup \{Q'_i \mid i \in \mathbb{N}_0 \text{ and } n_i = 1\} = Q'_0 = B_2(0), \\
Q^{(2)} &= \bigcup \{Q'_i \mid i \in \mathbb{N}_0 \text{ and } n_i = 2\} = \bigcup_{j \in \mathbb{N}} Q'_j = B_4(0) \setminus \overline{B_{1/4}}(0).
\end{aligned}$$

But the prerequisites of the current proposition include the assumptions $\widehat{\gamma_1^{(0)}}(\xi) = \widehat{\varphi}(\xi) \neq 0$ for all $\xi \in \overline{Q^{(1)}}$ and $\widehat{\gamma_2^{(0)}}(\xi) = \widehat{\psi}(\xi) \neq 0$ for all $\xi \in \overline{Q^{(2)}}$. By continuity of $\widehat{\varphi}, \widehat{\psi}$ and by compactness of $\overline{Q^{(1)}}, \overline{Q^{(2)}}$, we thus see that all assumptions of Lemma 3.7 are satisfied. Consequently, the family $\Gamma = (\gamma_i)_{i \in I}$ satisfies Assumption 3.6 and there is a constant $\Omega_3 = \Omega_3(\mathcal{B}, \varphi, \psi, p_0, \mu_0, d) > 0$ satisfying $\Omega_2^{(p,K)} \leq \Omega_3$ for all $K \leq \mu_0$ and $p \geq p_0$, with $\Omega_2^{(p,K)}$ as in Assumption 3.6. Recall that in our case, we indeed have $K = |\mu| \leq \mu_0$.

In view of the assumptions of the proposition, it is now not hard to see that all prerequisites for Corollary 6.6 are satisfied. Hence, that corollary implies that $\Gamma^{(\delta)}$ forms a Banach frame (in the sense of Theorem 4.7) for $\mathcal{B}_{s,\mu}^{p,q}(\mathbb{R}^d) = \mathcal{D}\left(\mathcal{Q}, L_{v(\mu)}^p, \ell_{(2^{js})_j}^q\right)$, as soon as $0 < \delta \leq \delta_{00}$, where (cf. Lemma 4.6 for the definition of F_0 and the estimate for $\|F_0\|$ used here)

$$\delta_{00} = \frac{1}{1 + 2\|F_0\|^2}$$

and, with $w = (2^{js})_{j \in \mathbb{N}_0}$,

$$\begin{aligned}
\|F_0\| &\leq 2^{\frac{1}{q}} C_{\mathcal{B}, \Phi, v_0, p}^2 \cdot \|\Gamma_{\mathcal{B}}\|_{\ell_w^q \rightarrow \ell_w^q}^2 \cdot \left(\|\vec{A}\|^{\max\{1, \frac{1}{p}\}} + \|\vec{B}\|^{\max\{1, \frac{1}{p}\}} \right) \cdot C_3 \\
&\quad (\text{Corollary 6.6}) \leq 2^{\frac{1}{q_0}} C_{\mathcal{B}, \Phi, v_0, p}^2 \cdot \|\Gamma_{\mathcal{B}}\|_{\ell_w^q \rightarrow \ell_w^q}^2 \cdot 4H_7 \cdot C_3 C_4 \\
&\quad (\text{eq. (1.15)}) \leq 4 \cdot 2^{\frac{1}{q_0}} C_{\mathcal{B}, \Phi, v_0, p}^2 \cdot \left[C_{\mathcal{B}, (2^{js})_{j \in \mathbb{N}_0}} \cdot N_{\mathcal{B}}^{1+\frac{1}{q}} \right]^2 \cdot H_7 \cdot C_3 C_4 \\
&\leq 4 \cdot 2^{\frac{1}{q_0}} C_{\mathcal{B}, \Phi, v_0, p}^2 \cdot \left[C_{\mathcal{B}, (2^j)_{j \in \mathbb{N}_0}}^{|s|} \cdot N_{\mathcal{B}}^{1+\frac{1}{q_0}} \right]^2 \cdot H_7 \cdot C_3 C_4 \\
&\leq 4 \cdot 2^{\frac{1}{q_0}} C_{\mathcal{B}, \Phi, v_0, p}^2 \cdot \left[C_{\mathcal{B}, (2^j)_{j \in \mathbb{N}_0}}^{\max\{s_1, -s_0\}} \cdot N_{\mathcal{B}}^{1+\frac{1}{q_0}} \right]^2 \cdot H_7 \cdot C_3 C_4,
\end{aligned}$$

where

$$C_3 = \begin{cases} \frac{(2^{16} \cdot 768 / d^{\frac{3}{2}})^{\frac{d}{p}}}{2^{42} \cdot 12^d \cdot d^{16}} \cdot \left(2^{52} \cdot d^{\frac{25}{2}} \cdot \tilde{N}^3\right)^{\tilde{N}+1} \cdot N_{\mathcal{B}}^{2(\frac{1}{p}-1)} (1 + R_{\mathcal{B}} C_{\mathcal{B}})^{d(\frac{4}{p}-1)} \cdot \Omega_0^{13K} \Omega_1^{13} \Omega_2^{(p,K)}, & \text{if } p < 1, \\ \frac{1}{\sqrt{d} \cdot 2^{12+6\lceil K \rceil}} \cdot \left(2^{17} \cdot d^{5/2} \cdot \tilde{N}\right)^{\lceil K \rceil + d + 2} \cdot (1 + R_{\mathcal{B}})^d \cdot \Omega_0^{3K} \Omega_1^3 \Omega_2^{(p,K)}, & \text{if } p \geq 1, \end{cases}$$

with $\tilde{N} = \left\lceil K + \frac{d+1}{\min\{1,p\}} \right\rceil \leq \left\lceil \mu_0 + \frac{d+1}{p_0} \right\rceil$ and

$$C_4 = \Omega_0^K \Omega_1 \cdot d^{1/\min\{1,p\}} \cdot (4 \cdot d)^{1+2\lceil K + \frac{d+\varepsilon}{\min\{1,p\}} \rceil} \cdot \left(\frac{sd}{\varepsilon}\right)^{1/\min\{1,p\}} \cdot \max_{|\alpha| \leq \lceil K + \frac{d+\varepsilon}{\min\{1,p\}} \rceil} C^{(\alpha)},$$

where the constants $C^{(\alpha)} = C^{(\alpha)}(\Phi)$ are as in Assumption 6.1.

To establish that δ_0 can be chosen independently of p, q, s, μ , it thus suffices to estimate $C_3 C_4$ and $C_{\mathcal{B}, \Phi, v_0, p}$ independently of these quantities. But above, we estimated $\Omega_2^{(p,K)} \leq \Omega_3$ with Ω_3 independent of p, q, s, μ . Since we also have $K = |\mu| \leq \mu_0$ and $0 \leq \frac{1}{p} \leq \frac{1}{p_0}$, as well as $\Omega_0 = 1$ and $\Omega_1 = 2^{|\mu|} \leq 2^{\mu_0}$, it is straightforward to see that C_3 can be estimated independently of p, q, s, μ . The same arguments also allow us to estimate C_4 independently of these quantities.

Finally, Corollary 6.5 shows that there is a suitable $\varrho \in C_c^\infty(\mathbb{R}^d)$ (depending only on \mathcal{B}) satisfying

$$C_{\mathcal{B}, \Phi, v_0, p} \leq \Omega_0^K \Omega_1 \cdot (4 \cdot d)^{1+2\lceil K + \frac{d+\varepsilon}{p} \rceil} \cdot \left(\frac{sd}{\varepsilon}\right)^{1/p} \cdot 2^{\lceil K + \frac{d+\varepsilon}{p} \rceil} \cdot \lambda_d(Q) \cdot \max_{|\alpha| \leq \lceil K + \frac{d+\varepsilon}{p} \rceil} \|\partial^\alpha \varrho\|_{\sup} \cdot \max_{|\alpha| \leq \lceil K + \frac{d+\varepsilon}{p} \rceil} C^{(\alpha)},$$

where $Q := \overline{\bigcup_{i \in \mathbb{N}_0} Q'_i} \subset \overline{B_4}(0)$. As above, since $0 \leq \frac{1}{p} \leq \frac{1}{p_0}$ and $K = |\mu| \leq \mu_0$, it is then not hard to see that $C_{\mathcal{B}, \Phi, v_0, p}$ can be estimated independently of p, q, s, μ .

It remains to show that the sequence space $\mathcal{C}_{p,q,s,\mu}$ is identical to the coefficient space $\ell^q_{\left(|\det T_i|^{\frac{1}{2}-\frac{1}{p}} \cdot w_i\right)_{i \in I}} \left([C_i^{(\delta)}]_{i \in I}\right)$

mentioned in Theorem 4.7. To this end, recall from equation (4.2) that $C_j^{(\delta)} = \ell^p_{v(j,\delta)}(\mathbb{Z}^d)$ with $v = v^{(\mu)}$ and

$$\begin{aligned} v_k^{(j,\delta)} &= v^{(\mu)}(\delta \cdot T_j^{-T} k) \\ &= (1 + |\delta \cdot T_j^{-T} k|)^\mu \\ &= (1 + |\delta \cdot k / 2^j|)^\mu. \end{aligned}$$

But since $0 < \delta \leq 1$, we have $\delta \cdot (1 + |k/2^j|) \leq 1 + |\delta \cdot \frac{k}{2^j}| \leq 1 + |k/2^j|$, which implies

$$\delta^{\mu_0} \cdot (1 + |k/2^j|)^\mu \leq \delta^{|\mu|} \cdot (1 + |k/2^j|)^\mu \leq v_k^{(j,\delta)} \leq \delta^{-|\mu|} \cdot (1 + |k/2^j|)^\mu \leq \delta^{-\mu_0} \cdot (1 + |k/2^j|)^\mu$$

for all $k \in \mathbb{Z}^d$, $j \in \mathbb{N}_0$ and $0 < \delta \leq 1$. Finally, since $w_i = 2^{s_i}$ and $|\det T_i| = 2^{i \cdot d}$ for $i \in \mathbb{N}_0$, we see

$$|\det T_i|^{\frac{1}{2}-\frac{1}{p}} \cdot w_i = 2^{i(s+d(\frac{1}{2}-\frac{1}{p}))}.$$

Taken together, these considerations easily show $\ell^q_{\left(|\det T_i|^{\frac{1}{2}-\frac{1}{p}} \cdot w_i\right)_{i \in I}} \left([C_i^{(\delta)}]_{i \in I}\right) = \mathcal{C}_{p,q,s,\mu}$ with equivalent quasi-norms.

Here, the implicit constant is allowed to depend on δ . \square

Next, we derive concrete conditions on φ, ψ which ensure that the generated wavelet system yields atomic decompositions for the (weighted) Besov spaces $\mathcal{B}_{s,\mu}^{p,q}(\mathbb{R}^d)$.

Proposition 8.4. *Let $p_0, q_0 \in (0, 1]$, $\varepsilon > 0$, $\mu_0 \geq 0$ and $-\infty < s_0 \leq s_1 < \infty$. Assume that $\varphi, \psi \in L^1(\mathbb{R}^d)$ satisfy the following conditions:*

- (1) *We have $\|\varphi\|_{K_{00}} < \infty$ and $\|\psi\|_{K_{00}} < \infty$ for $K_{00} := \mu_0 + \frac{d}{p_0} + 1$, where $\|g\|_M = \sup_{x \in \mathbb{R}^d} (1 + |x|)^M \cdot |g(x)|$.*
- (2) *We have $\widehat{\varphi}, \widehat{\psi} \in C^\infty(\mathbb{R}^d)$, with all partial derivatives of $\widehat{\varphi}, \widehat{\psi}$ being polynomially bounded.*
- (3) *We have $\widehat{\varphi}(\xi) \neq 0$ for all $\xi \in \overline{B_2}(0)$ and $\widehat{\psi}(\xi) \neq 0$ for all $\xi \in \overline{B_4}(0) \setminus B_{1/4}(0)$.*
- (4) *We have*

$$\begin{aligned} |\partial^\alpha \widehat{\varphi}(\xi)| &\leq G_1 \cdot (1 + |\xi|)^{-L}, \\ |\partial^\alpha \widehat{\psi}(\xi)| &\leq G_2 \cdot (1 + |\xi|)^{-L_1} \cdot \min\{1, |\xi|^{L_2}\} \end{aligned}$$

for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq N_0 := \left\lceil \mu_0 + \frac{d+\varepsilon}{p_0} \right\rceil$, all $\xi \in \mathbb{R}^d$, suitable $G_1, G_2 > 0$ and certain $L, L_1 \geq 2d+1+2\varepsilon$ and $L_2 \geq 0$ which furthermore satisfy

$$L > s_1 + \kappa + d + 1 + \varepsilon, \quad L_1 > s_1 + \kappa + d + 1 + \varepsilon, \quad \text{and} \quad L_2 > \vartheta_0 d - s_0$$

for

$$\vartheta_0 := \begin{cases} 0, & \text{if } p_0 = 1 \\ \frac{1}{p_0} - 1, & \text{if } p_0 \in (0, 1), \end{cases} \quad \text{and} \quad \kappa := \begin{cases} \lceil \mu_0 + d + \varepsilon \rceil, & \text{if } p_0 = 1, \\ d + \mu_0 + \left\lceil \mu_0 + \frac{d+\varepsilon}{p_0} \right\rceil, & \text{if } p_0 \in (0, 1). \end{cases}$$

Then there is some $\delta_0 = \delta_0(d, p_0, q_0, \varepsilon, \mu_0, s_0, s_1, \varphi, \psi) > 0$ such that for each $0 < \delta \leq \delta_0$, the family

$$\Gamma^{(\delta)} = \left(2^{j \frac{d}{2}} \cdot \gamma_j(2^j \bullet - \delta k) \right)_{j \in \mathbb{N}_0, k \in \mathbb{Z}^d}, \quad \text{with} \quad \gamma_0 := \varphi \quad \text{and} \quad \gamma_j := \psi \text{ for } j \in \mathbb{N},$$

forms an atomic decomposition for $\mathcal{B}_{s, \mu}^{p, q}(\mathbb{R}^d)$, for arbitrary $p, q \in (0, \infty]$ and $s, \mu \in \mathbb{R}$ satisfying $p \geq p_0$, $q \geq q_0$ as well as $s_0 \leq s \leq s_1$ and $|\mu| \leq \mu_0$.

Precisely, this means the following: With the space $\mathcal{C}_{p, q, s, \mu} \leq \mathbb{C}^{\mathbb{N}_0 \times \mathbb{Z}^d}$ as in Proposition 8.3, the following are true:

(1) **The synthesis map**

$$S^{(\delta)} : \mathcal{C}_{p, q, s, \mu} \rightarrow \mathcal{B}_{s, \mu}^{p, q}(\mathbb{R}^d), (c_k^{(i)})_{i \in \mathbb{N}_0, k \in \mathbb{Z}^d} \mapsto \sum_{i \in \mathbb{N}_0} \sum_{k \in \mathbb{Z}^d} \left[c_k^{(i)} \cdot 2^{i \frac{d}{2}} \cdot \gamma_i(2^i \bullet - \delta k) \right]$$

is well-defined and bounded for each $0 < \delta \leq 1$.

Convergence of the series has to be understood as described in the remark after Theorem 5.6.

(2) For $0 < \delta \leq \delta_0$, there is a bounded linear **coefficient map**

$$C^{(\delta)} : \mathcal{B}_{s, \mu}^{p, q}(\mathbb{R}^d) \rightarrow \mathcal{C}_{p, q, s, \mu}$$

satisfying $S^{(\delta)} \circ C^{(\delta)} = \text{id}_{\mathcal{B}_{s, \mu}^{p, q}(\mathbb{R}^d)}$. Furthermore, the action of $C^{(\delta)}$ on a given $f \in \mathcal{B}_{s, \mu}^{p, q}(\mathbb{R}^d)$ is independent of the precise choice of p, q, s, μ . \blacktriangleleft

Proof. Define $\tilde{L} := L - (d + 1 + \varepsilon)$ and $\tilde{L}_1 := L_1 - (d + 1 + \varepsilon)$. Now, an application of Lemma 6.9 (with $\gamma = \psi$, $N = N_0 \geq \left\lceil \frac{d+\varepsilon}{p_0} \right\rceil \geq \lceil d + \varepsilon \rceil \geq d + 1$ and $\varrho(\xi) := G_2 \cdot (1 + |\xi|)^{-\tilde{L}_1} \cdot \min \left\{ 1, |\xi|^{L_2} \right\}$, where $\varrho \in L^1(\mathbb{R}^d)$ since $\tilde{L}_1 \geq d + \varepsilon$) yields functions $\psi_1, \psi_2 \in L^1(\mathbb{R}^d)$ with the following properties:

- (1) We have $\psi = \psi_1 * \psi_2$.
- (2) We have $\psi_2 \in C^1(\mathbb{R}^d)$ with $H_1^{(M)} := \|\psi_2\|_M + \|\nabla \psi_2\|_M < \infty$ for all $M \in \mathbb{N}_0$.
- (3) We have $\widehat{\psi_1}, \widehat{\psi_2} \in C^\infty(\mathbb{R}^d)$, where all partial derivatives of these functions are polynomially bounded.
- (4) We have $\|\psi_1\|_{N_0} < \infty$ and $\|\psi\|_{N_0} < \infty$. In particular, since $N_0 \geq \mu_0 + \frac{d+\varepsilon}{p_0} \geq \mu_0 + d + \varepsilon$, we have $\psi_1 \in L_{(1+|\bullet|)^{\mu_0+d+\varepsilon}}^\infty(\mathbb{R}^d) \hookrightarrow L_{(1+|\bullet|)^{\mu_0}}^1(\mathbb{R}^d) \hookrightarrow L_{(1+|\bullet|)^K}^1(\mathbb{R}^d)$ for all $K = |\mu| \leq \mu_0$.
- (5) We have

$$\begin{aligned} \left| \partial^\alpha \widehat{\psi_1}(\xi) \right| &\leq 2^{1+d+4N_0} \cdot N_0! \cdot (1+d)^{N_0} \cdot \varrho(\xi) \\ &\leq H_2 \cdot (1+|\xi|)^{-\tilde{L}_1} \cdot \min \left\{ 1, |\xi|^{L_2} \right\} \end{aligned} \tag{8.8}$$

with $H_2 := G_2 \cdot 2^{1+d+4N_0} \cdot N_0! \cdot (1+d)^{N_0}$ for all $\xi \in \mathbb{R}^d$ and all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq N_0$.

Likewise, another application of Lemma 6.9 (this time with $\gamma = \varphi$, $N = N_0 \geq d + 1$ and $\varrho(\xi) := G_1 \cdot (1 + |\xi|)^{-\tilde{L}}$, where $\varrho \in L^1(\mathbb{R}^d)$, since $\tilde{L} \geq d + \varepsilon$) yields certain functions $\varphi_1, \varphi_2 \in L^1(\mathbb{R}^d)$ with the following properties:

- (1) We have $\varphi = \varphi_1 * \varphi_2$.
- (2) We have $\varphi_2 \in C^1(\mathbb{R}^d)$ with $H_3^{(M)} := \|\varphi_2\|_M + \|\nabla \varphi_2\|_M < \infty$ for all $M \in \mathbb{N}_0$.
- (3) We have $\widehat{\varphi_1}, \widehat{\varphi_2} \in C^\infty(\mathbb{R}^d)$, where all partial derivatives of these functions are polynomially bounded.
- (4) We have $\|\varphi_1\|_{N_0} < \infty$ and $\|\varphi\|_{N_0} < \infty$. As for ψ_1 , this implies $\varphi_1 \in L_{(1+|\bullet|)^{\mu_0}}^1(\mathbb{R}^d) \hookrightarrow L_{(1+|\bullet|)^K}^1(\mathbb{R}^d)$ for all $K = |\mu| \leq \mu_0$.
- (5) We have

$$\begin{aligned} \left| \partial^\alpha \widehat{\varphi_1}(\xi) \right| &\leq 2^{1+d+4N_0} \cdot N_0! \cdot (1+d)^{N_0} \cdot \varrho(\xi) \\ &\leq H_4 \cdot (1+|\xi|)^{-\tilde{L}} \end{aligned} \tag{8.9}$$

with $H_4 := G_1 \cdot 2^{1+d+4N_0} \cdot N_0! \cdot (1+d)^{N_0}$ for all $\xi \in \mathbb{R}^d$ and all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq N_0$.

Now, set $\gamma_0 := \varphi$ and $\gamma_{0,\ell} := \varphi_\ell$, as well as $\gamma_j := \psi$ and $\gamma_{j,\ell} := \psi_\ell$ for $\ell \in \{1, 2\}$ and $j \in \mathbb{N}$.

As a further preparation, set $\gamma_1^{(0)} := \varphi$ and $\gamma_2^{(0)} := \psi$, as well as $n_0 := 1$ and $n_j := 2$ for $j \in \mathbb{N}$, so that $\gamma_j = \gamma_{n_j}^{(0)}$ for all $j \in \mathbb{N}_0$. Then, in the notation of Lemma 3.7, we have $Q^{(1)} = B_2(0)$ and $Q^{(2)} = B_4(0) \setminus \overline{B_{1/4}}(0)$, cf. the proof of Proposition 8.3. Exactly as in that proof, we see that all prerequisites of Lemma 3.7 are satisfied, so that the family $\Gamma = (\gamma_i)_{i \in \mathbb{N}_0}$ satisfies Assumption 3.6, where we furthermore have $\Omega_2^{(p,K)} \leq \Omega_3$ for all $K \leq \mu_0$ and all $p \geq p_0$, for a suitable constant $\Omega_3 = \Omega_3(\mathcal{B}, \varphi, \psi, p_0, \mu_0, d) > 0$. Observe that indeed $K = |\mu| \leq \mu_0$ in the cases which are of interest to us.

Now, let p, q, s, μ be as in the statement of the proposition. We want to verify the prerequisites of Corollary 6.7 for the choices which we just made. For most of these assumptions, this is not hard:

- (1) All $\gamma_i, \gamma_{i,1}, \gamma_{i,2}$ are measurable functions, as required.
- (2) Since $K = |\mu| \leq \mu_0$, we have $\gamma_{i,1} = \psi_1 \in L^1_{(1+|\bullet|)^\kappa}(\mathbb{R}^d)$ for any $i \in \mathbb{N}$ and also $\gamma_{0,1} = \varphi_1 \in L^1_{(1+|\bullet|)^\kappa}(\mathbb{R}^d)$, as seen above.
- (3) We have $\gamma_{i,2} = \psi_2 \in C^1(\mathbb{R}^d)$ for any $i \in \mathbb{N}$ and also $\gamma_{0,2} = \varphi_2 \in C^1(\mathbb{R}^d)$, as noted above.
- (4) With $K_0 := K + \frac{d}{\min\{1,p\}} + 1 \leq \mu_0 + \frac{d}{p_0} + 1 = K_{00}$, we have

$$\begin{aligned} \Omega_4^{(p,K)} &= \sup_{i \in I} \|\gamma_{i,2}\|_{K_0} + \sup_{i \in I} \|\nabla \gamma_{i,2}\|_{K_0} \\ &\leq \max\{\|\varphi_2\|_{K_{00}}, \|\psi_2\|_{K_{00}}\} + \max\{\|\nabla \varphi_2\|_{K_{00}}, \|\nabla \psi_2\|_{K_{00}}\} \\ &\leq H_1^{(\lceil K_{00} \rceil)} + H_3^{(\lceil K_{00} \rceil)} =: H_5 < \infty. \end{aligned} \tag{8.10}$$

- (5) We have $\|\gamma_i\|_{K_0} \leq \|\gamma_i\|_{K_{00}} < \infty$ for all $i \in \mathbb{N}_0$, since $\|\varphi\|_{K_{00}} < \infty$ and $\|\psi\|_{K_{00}} < \infty$ by assumption.
- (6) We have $\gamma_i = \psi = \psi_1 * \psi_2 = \gamma_{i,1} * \gamma_{i,2}$ for all $i \in \mathbb{N}$ and likewise $\gamma_0 = \varphi = \varphi_1 * \varphi_2 = \gamma_{0,1} * \gamma_{0,2}$.
- (7) We have $\widehat{\gamma_{i,1}}, \widehat{\gamma_{i,2}} \in C^\infty(\mathbb{R}^d)$ for all $i \in \mathbb{N}_0$, and all partial derivatives of these functions are polynomially bounded.
- (8) As we showed above, the family $\Gamma = (\gamma_i)_{i \in \mathbb{N}_0}$ satisfies Assumption 3.6.

Hence, the only prerequisite of Corollary 6.7 which still needs to be verified is that

$$K_1 := \sup_{i \in \mathbb{N}_0} \sum_{j \in \mathbb{N}_0} N_{i,j} < \infty \quad \text{and} \quad K_2 := \sup_{j \in \mathbb{N}_0} \sum_{i \in \mathbb{N}_0} N_{i,j} < \infty,$$

where

$$N_{i,j} := \left(\frac{w_i}{w_j} \cdot (|\det T_j| / |\det T_i|)^\vartheta \right)^\tau \cdot (1 + \|T_j^{-1} T_i\|)^\sigma \cdot \left(|\det T_i|^{-1} \cdot \int_{Q_i} \max_{|\alpha| \leq N} |(\partial^\alpha \widehat{\gamma_{j,1}})(S_j^{-1} \xi)| \, d\xi \right)^\tau,$$

with

$$\begin{aligned} N &= \left\lceil K + \frac{d + \varepsilon}{\min\{1, p\}} \right\rceil, \\ \tau &= \min\{1, p, q\}, \\ \sigma &= \begin{cases} \min\{1, q\} \cdot \lceil K + d + \varepsilon \rceil, & \text{if } p \in [1, \infty], \\ \min\{p, q\} \cdot \left(\frac{d}{p} + K + \left\lceil K + \frac{d + \varepsilon}{p} \right\rceil \right), & \text{if } p \in (0, 1), \end{cases} \\ \vartheta &= \begin{cases} 0, & \text{if } p \in [1, \infty], \\ \frac{1}{p} - 1, & \text{if } p \in (0, 1). \end{cases} \end{aligned}$$

To prove this, we begin with several auxiliary observations: First of all, we observe $N \leq N_0$. Furthermore, we have $\vartheta \leq \vartheta_0$, since $p_0 = 1$ implies $p \in [1, \infty]$. Finally, we also have

$$\begin{aligned} \frac{\sigma}{\tau} - \vartheta d &= \begin{cases} \lceil K + d + \varepsilon \rceil, & \text{if } p \in [1, \infty], \\ \left\lfloor \frac{d}{p} + K + \left\lceil K + \frac{d + \varepsilon}{p} \right\rceil \right\rfloor - d \left(\frac{1}{p} - 1 \right), & \text{if } p \in (0, 1) \end{cases} \\ &\leq \begin{cases} \lceil \mu_0 + d + \varepsilon \rceil, & \text{if } p_0 = 1, \\ d + \mu_0 + \left\lceil \mu_0 + \frac{d + \varepsilon}{p_0} \right\rceil, & \text{if } p_0 \in (0, 1) \end{cases} \\ &= \kappa, \end{aligned} \tag{8.11}$$

where we used that $K = |\mu| \leq \mu_0$ and also that $p_0 = 1$ entails $p \in [1, \infty]$.

We divide our estimates of $N_{i,j}$ into two main cases. The first case is $i \in \mathbb{N}$. Here, we recall from the proof of Proposition 8.3 (cf. equation (8.4)) for $Q_i = \{\xi \in \mathbb{R}^d \mid 2^{i-2} < |\xi| < 2^{i+2}\}$ that $\lambda_d(Q_i) \leq 2^{i \cdot d} \cdot 4^d v_d$. Since we also have $N \leq N_0$, we conclude

$$N_{i,j} \leq \left[2^{s(i-j)+\vartheta d(j-i)} \right]^\tau \cdot (1 + 2^{i-j})^\sigma \cdot \left(4^d v_d \cdot \sup_{2^{i-2} < |\xi| < 2^{i+2}} \max_{|\alpha| \leq N_0} |(\partial^\alpha \widehat{\gamma_{j,1}})(\xi/2^j)| \right)^\tau. \quad (8.12)$$

To further estimate this term, we consider two subcases:

Case 1: We have $j \in \mathbb{N}$. Here, we again consider two subcases:

(1) We have $i \leq j$. For $2^{i-2} < |\xi| < 2^{i+2}$, equation (8.8) implies because of $\widetilde{L}_1 \geq 0$ and $L_2 \geq 0$ that

$$\begin{aligned} \max_{|\alpha| \leq N_0} |(\partial^\alpha \widehat{\gamma_{j,1}})(\xi/2^j)| &= \max_{|\alpha| \leq N_0} |(\partial^\alpha \widehat{\psi_1})(\xi/2^j)| \leq H_2 \cdot \min \left\{ 1, |\xi/2^j|^{L_2} \right\} \\ &\leq H_2 \cdot 4^{L_2} \cdot 2^{L_2(i-j)} \\ &= 4^{L_2} H_2 \cdot 2^{-L_2|i-j|}. \end{aligned}$$

In combination with equation (8.12), we get

$$N_{i,j} \leq 2^\sigma \cdot (4^{d+L_2} v_d \cdot H_2)^\tau \cdot 2^{-\tau|i-j|(L_2+s-\vartheta d)}.$$

(2) We have $j \leq i$. Here, equation (8.8) implies for $2^{i-2} < |\xi| < 2^{i+2}$ that

$$\begin{aligned} \max_{|\alpha| \leq N_0} |(\partial^\alpha \widehat{\gamma_{j,1}})(\xi/2^j)| &= \max_{|\alpha| \leq N_0} |(\partial^\alpha \widehat{\psi_1})(\xi/2^j)| \leq H_2 \cdot (1 + |\xi/2^j|)^{-\widetilde{L}_1} \\ &\leq H_2 \cdot (2^{i-j}/4)^{-\widetilde{L}_1} \\ &= 4^{\widetilde{L}_1} H_2 \cdot 2^{-\widetilde{L}_1|i-j|}. \end{aligned}$$

In combination with equation (8.12), we get

$$N_{i,j} \leq 2^\sigma \cdot (4^{d+\widetilde{L}_1} v_d \cdot H_2)^\tau \cdot 2^{-\tau|i-j|(-\frac{\sigma}{\tau}-s+\vartheta d+\widetilde{L}_1)}.$$

Case 2: We have $j = 0$. Here, equation (8.9) shows for $2^{i-2} < |\xi| < 2^{i+2}$ that

$$\begin{aligned} \max_{|\alpha| \leq N_0} |(\partial^\alpha \widehat{\gamma_{j,1}})(\xi/2^j)| &= \max_{|\alpha| \leq N_0} |\partial^\alpha \widehat{\varphi_1}(\xi)| \leq H_4 \cdot (1 + |\xi|)^{-\widetilde{L}} \\ &\leq H_4 \cdot (2^i/4)^{-\widetilde{L}} \\ &= 4^{\widetilde{L}} H_4 \cdot 2^{-i\widetilde{L}}. \end{aligned}$$

In combination with equation (8.12), this implies

$$N_{i,j} \leq 2^\sigma \cdot (4^{d+\widetilde{L}} v_d \cdot H_4)^\tau \cdot 2^{-\tau i(\widetilde{L}-s+\vartheta d-\frac{\sigma}{\tau})} = 2^\sigma \cdot (4^{d+\widetilde{L}} v_d \cdot H_4)^\tau \cdot 2^{-\tau|i-j|(\widetilde{L}-s+\vartheta d-\frac{\sigma}{\tau})}.$$

These are the desired estimates for the case $i \in \mathbb{N}$.

If otherwise $i = 0$, so that $Q_i = T_0 Q'_0 + b_0 = Q'_0 = B_2(0)$, estimate (8.12) takes the modified form

$$\begin{aligned} N_{i,j} &\leq 2^{\tau j(\vartheta d-s)} \cdot (1 + 2^{-j})^\sigma \cdot \left(\int_{B_2(0)} \max_{|\alpha| \leq N} |(\partial^\alpha \widehat{\gamma_{j,1}})(S_j^{-1}\xi)| \, d\xi \right)^\tau \\ &\leq 2^\sigma \cdot [\lambda_d(B_2(0))]^\tau \cdot 2^{\tau j(\vartheta d-s)} \cdot \sup_{|\xi| < 2} \left(\max_{|\alpha| \leq N_0} |(\partial^\alpha \widehat{\gamma_{j,1}})(\xi/2^j)| \right)^\tau. \end{aligned} \quad (8.13)$$

To further estimate this quantity, we again distinguish two cases:

Case 1: We have $j \in \mathbb{N}$. In this case, equation (8.8) shows for $|\xi| < 2$ that

$$\begin{aligned} \max_{|\alpha| \leq N_0} |(\partial^\alpha \widehat{\gamma_{j,1}})(\xi/2^j)| &= \max_{|\alpha| \leq N_0} |(\partial^\alpha \widehat{\psi_1})(\xi/2^j)| \leq H_2 \cdot \min \left\{ 1, |\xi/2^j|^{L_2} \right\} \\ &\leq 2^{L_2} H_2 \cdot 2^{-jL_2}. \end{aligned}$$

In combination with equation (8.13), we get

$$\begin{aligned} N_{i,j} &\leq 2^\sigma \cdot [2^{L_2+d} v_d \cdot H_2]^\tau \cdot 2^{\tau j(\vartheta d - s - L_2)} \\ &= 2^\sigma \cdot [2^{L_2+d} v_d \cdot H_2]^\tau \cdot 2^{-\tau|i-j|(L_2+s-\vartheta d)}. \end{aligned}$$

Case 2: We have $j = 0$. In this case, equation (8.9) shows for $|\xi| < 2$ that

$$\max_{|\alpha| \leq N_0} |(\partial^\alpha \widehat{\gamma_{j,1}})(\xi/2^j)| = \max_{|\alpha| \leq N_0} |(\partial^\alpha \widehat{\varphi_1})(\xi)| \leq H_4,$$

since $\widetilde{L} \geq 0$. In combination with equation (8.13), this yields

$$N_{i,j} \leq 2^\sigma \cdot [2^d v_d \cdot H_4]^\tau = 2^\sigma \cdot [2^d v_d \cdot H_4]^\tau \cdot 2^{-\tau|i-j|\zeta}$$

for arbitrary $\zeta \in \mathbb{R}$, since $|i-j| = 0$.

All in all, our considerations show that there is some constant H_6 which is independent of p, q, s, μ , such that

$$N_{i,j} \leq 2^\sigma \cdot H_6^\tau \cdot 2^{-\tau|i-j|\lambda} \quad \text{where} \quad \lambda := \min \left\{ 1, L_2 + s - \vartheta d, \widetilde{L} - s + \vartheta d - \frac{\sigma}{\tau}, \widetilde{L}_1 - s + \vartheta d - \frac{\sigma}{\tau} \right\}.$$

But in view of equation (8.11) and since $s_0 \leq s \leq s_1$, as well as $\vartheta \leq \vartheta_0$, we have

$$\lambda \geq \min \left\{ 1, L_2 + s_0 - \vartheta_0 d, \widetilde{L} - s_1 - \kappa, \widetilde{L}_1 - s_1 - \kappa \right\} =: \lambda_0 > 0,$$

by our assumptions regarding L, L_1, L_2 . Note that λ_0 is independent of p, q, s, μ .

All in all, we thus get, for K_1, K_2 as in Corollary 6.7,

$$\begin{aligned} K_1^{1/\tau} &= \sup_{i \in \mathbb{N}_0} \left(\sum_{j \in \mathbb{N}_0} N_{i,j} \right)^{1/\tau} \leq 2^{\sigma/\tau} \cdot H_6 \cdot \sup_{i \in \mathbb{N}_0} \left(\sum_{j \in \mathbb{N}_0} 2^{-\tau\lambda|i-j|} \right)^{1/\tau} \\ &\quad \left(\text{since } \ell^{\tau_0} \hookrightarrow \ell^\tau \text{ is norm-decreasing for } \tau_0 := \min\{1, p_0, q_0\} \leq \tau \right) \leq 2^{\sigma/\tau} \cdot H_6 \cdot \sup_{i \in \mathbb{N}_0} \left(\sum_{j \in \mathbb{N}_0} 2^{-\tau_0\lambda|i-j|} \right)^{1/\tau_0} \\ &\quad \left(\text{since } \lambda \geq \lambda_0 \text{ and with } \ell = |i-j| \right) \leq 2^{\sigma/\tau} \cdot H_6 \cdot \left(\sum_{\ell \in \mathbb{Z}} 2^{-\tau_0\lambda_0|\ell|} \right)^{1/\tau_0} \\ &\leq 2^{\frac{\sigma}{\tau} + \frac{1}{\tau_0}} \cdot H_6 \cdot \left(\sum_{\ell=0}^{\infty} 2^{-\tau_0\lambda_0\ell} \right)^{1/\tau_0} \\ &\quad \left(\text{since } \frac{\sigma}{\tau} \leq \kappa + \vartheta d \leq \kappa + \vartheta_0 d \text{ by eq. (8.11)} \right) \leq 2^{\kappa + \vartheta_0 d + \frac{1}{\tau_0}} \cdot H_6 \cdot \left(\frac{1}{1 - 2^{-\tau_0\lambda_0}} \right)^{1/\tau_0} =: H_7, \end{aligned}$$

where H_7 is independent of p, q, s, μ . Precisely the same estimate also yields $K_2^{1/\tau} \leq H_7$.

All in all, we see that Corollary 6.7 is applicable, so that the operator \vec{C} from Assumption 5.1 satisfies $\|\vec{C}\|^{\max\{1, \frac{1}{p}\}} \leq \Omega \cdot (K_1^{1/\tau} + K_2^{1/\tau}) \leq 2\Omega H_7$ for $\Omega = \Omega_0^K \Omega_1 \cdot (4 \cdot d)^{1+2\lceil K + \frac{d+\varepsilon}{\min\{1, p\}} \rceil} \cdot \left(\frac{s_d}{\varepsilon}\right)^{1/\min\{1, p\}} \cdot \max_{|\alpha| \leq N} C^{(\alpha)}$, where the constants $C^{(\alpha)}$ are as in Assumption 6.1. Since $N \leq N_0$, $K = |\mu| \leq \mu_0$ and $\frac{1}{\min\{1, p\}} \leq \frac{1}{p_0}$, as well as $\Omega_0 = 1$ and $\Omega_1 = 2^{|\mu|} \leq 2^{\mu_0}$, it is not hard to see $\Omega \leq H_8$, where H_8 is independent of p, q, s, μ .

Corollary 6.7 ensures that Theorem 5.6 is applicable, i.e., the family $\Gamma^{(\delta)}$ from the statement of the current proposition is indeed an atomic decomposition for $\mathcal{B}_{s, \mu}^{p, q}(\mathbb{R}^d) = \mathcal{D}(\mathcal{B}, L_{v(\mu)}^p, \ell_{(2^{js})_{j \in \mathbb{N}_0}}^q)$ as soon as $0 < \delta \leq \min\{1, \delta_{00}\}$, where δ_{00} is defined by

$$\delta_{00}^{-1} := \begin{cases} \frac{2s_d}{\sqrt{d}} \cdot (2^{17} \cdot d^2 \cdot (K+2+d))^{K+d+3} \cdot (1+R_{\mathcal{B}})^{d+1} \cdot \Omega_0^{4K} \Omega_1^{4\Omega_2^{(p,K)}} \Omega_4^{(p,K)} \cdot \|\vec{C}\|, & \text{if } p \geq 1, \\ \frac{(2^{14}/d^{\frac{3}{2}})^{\frac{d}{p}}}{2^{45} \cdot d^{17}} \cdot \left(\frac{s_d}{p}\right)^{\frac{1}{p}} \left(2^{68} \cdot d^{14} \cdot \left(K+1+\frac{d+1}{p}\right)^3\right)^{K+2+\frac{d+1}{p}} \cdot (1+R_{\mathcal{B}})^{1+\frac{3d}{p}} \cdot \Omega_0^{16K} \Omega_1^{16\Omega_2^{(p,K)}} \Omega_4^{(p,K)} \cdot \|\vec{C}\|^{\frac{1}{p}}, & \text{if } p < 1. \end{cases}$$

The verification that the discrete sequence space from Theorem 5.6 coincides with $\mathcal{C}_{p, q, s, \mu}$ is exactly as in the proof of Proposition 8.3. Hence, to complete the proof, we only have to verify $\delta_{00}^{-1} \leq \delta_0^{-1}$, where $\delta_0 > 0$ is independent of p, q, s, μ .

But above, we showed $\Omega_2^{(p,K)} \leq \Omega_3$, with Ω_3 independent of p, q, s, μ , since $K = |\mu| \leq \mu_0$ and $p \geq p_0$. Furthermore, equation (8.10) shows $\Omega_4^{(p,K)} \leq H_5$ for $K = |\mu| \leq \mu_0$ and with H_5 independent of p, q, s, μ . Using these

estimates, the estimate for $\|\vec{C}\|$ from above, and the straightforward inequalities $0 \leq \frac{1}{p} \leq \frac{1}{p_0}$ and $K = |\mu| \leq \mu_0$, as well as $\Omega_0 = 1$ and $\Omega_2 = 2^{|\mu|} \leq 2^{\mu_0}$, we see that indeed $\delta_{00}^{-1} \leq \delta_0^{-1}$ for some $\delta_0 > 0$ independent of p, q, s, μ . \square

Remark 8.5. We close this section by showing that our results indeed imply the existence of compactly supported Banach frames and atomic decompositions for Besov spaces. Finally, we compare our results with the literature.

- The conditions in Propositions 8.3 and 8.4 are still not completely straightforward to verify. Thus, let $k, N \in \mathbb{N}$ and $L_1, L_2 \geq 0$ be arbitrary. We will show how one can construct a compactly supported function $\psi \in C_c^k(\mathbb{R}^d)$ satisfying $\widehat{\psi}(\xi) \neq 0$ for $\xi \in \overline{B_4}(0) \setminus B_{1/4}(0)$ as well as

$$\left| \partial^\alpha \widehat{\psi}(\xi) \right| \lesssim (1 + |\xi|)^{-L_1} \cdot \min \left\{ 1, |\xi|^{L_2} \right\} \quad \forall \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \leq N.$$

To this end, let $\ell := \max \{k, \lceil L_1 \rceil\}$ and $L_3 := \max \{0, \lceil (L_2 - N - 1)/2 \rceil\}$ and choose $\psi_0 \in C_c^{\ell+2(L_3+N+1)}(\mathbb{R}^d)$ with $\psi_0 \geq 0$ and $\psi_0 \not\equiv 0$. This implies $\widehat{\psi_0}(0) = \|\psi_0\|_{L^1} > 0$. By continuity of $\widehat{\psi_0}$, there is thus some $c_0 > 0$ and some $\varepsilon > 0$ satisfying $|\widehat{\psi_0}(\xi)| \geq c_0$ for $|\xi| \leq \varepsilon$.

Define $\psi_1 := \psi_0 \circ \frac{4}{\varepsilon} \text{id}$ and note $|\widehat{\psi_1}(\xi)| = (\varepsilon/4)^d \cdot |\widehat{\psi_0}(\frac{\varepsilon}{4}\xi)| \geq c_0 \cdot (\varepsilon/4)^d =: c_1$ as long as $|\xi| \leq 4$. Next, set

$$\psi := \Delta^{L_3+N+1} \psi_1 \in C_c^\ell(\mathbb{R}^d) \subset C_c^k(\mathbb{R}^d),$$

where $\Delta = \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}$ denotes the Laplace operator. An easy calculation using partial integration shows $\mathcal{F}[\Delta g](\xi) = -4\pi^2 \cdot |\xi|^2 \cdot \widehat{g}(\xi)$ for $g \in C_c^2(\mathbb{R}^d)$, so that we get

$$\widehat{\psi}(\xi) = (-4\pi^2)^{L_3+N+1} \cdot |\xi|^{2(L_3+N+1)} \cdot \widehat{\psi_1}(\xi) \quad \forall \xi \in \mathbb{R}^d. \quad (8.14)$$

In particular, $\widehat{\psi}(\xi) \in o(|\xi|^{2(L_3+N+1)})$ as $\xi \rightarrow 0$. Furthermore, since $\psi \in C_c^\ell(\mathbb{R}^d)$, we have $\widehat{\psi} \in C^\infty(\mathbb{R}^d)$, so that Lemma B.2 shows $\partial^\alpha \widehat{\psi}(\xi) \in o(|\xi|^{2(L_3+N+1)-|\alpha|}) \subset o(|\xi|^{2L_3+N+1}) \subset o(|\xi|^{L_2})$ as $\xi \rightarrow 0$, for $|\alpha| \leq N$. By continuity of $\partial^\alpha \widehat{\psi}$, this implies that there is a constant $C > 0$ satisfying

$$\left| \partial^\alpha \widehat{\psi}(\xi) \right| \leq C \cdot |\xi|^{L_2} \leq 2^{L_1} C \cdot (1 + |\xi|)^{-L_1} \cdot \min \left\{ 1, |\xi|^{L_2} \right\}$$

for all $|\xi| \leq 1$ and all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq N$.

Furthermore, since $\psi \in C_c^\ell(\mathbb{R}^d)$ and thus also $\psi^{(\alpha)} := [x \mapsto (-2\pi i x)^\alpha \cdot \psi(x)] \in C_c^\ell(\mathbb{R}^d)$, Lemma 6.3 and elementary properties of the Fourier transform imply

$$\left| \partial^\alpha \widehat{\psi}(\xi) \right| = \left| (\mathcal{F}^{-1} \psi^{(\alpha)})(-\xi) \right| \lesssim (1 + |\xi|)^{-\ell} \leq (1 + |\xi|)^{-L_1} = (1 + |\xi|)^{-L_1} \cdot \min \left\{ 1, |\xi|^{L_2} \right\} \quad \text{for } |\xi| \geq 1$$

for arbitrary $\alpha \in \mathbb{N}_0^d$. Hence, $\partial^\alpha \widehat{\psi}$ satisfies the desired decay properties.

Finally, note that eq. (8.14) also yields $|\widehat{\psi}(\xi)| = (4\pi^2)^{L_3+N+1} \cdot |\xi|^{2(L_3+N+1)} \cdot |\widehat{\psi_1}(\xi)| \geq c_1 \cdot (\pi^2/4)^{L_3+N+1} \geq c_1$ for all $\xi \in \overline{B_4}(0) \setminus B_{1/4}(0)$. We have thus constructed ψ as desired. Similarly, but easier, one can construct $\varphi \in C_c^k(\mathbb{R}^d)$ satisfying $\widehat{\varphi}(\xi) \neq 0$ for $\xi \in \overline{B_2}(0)$ and $|\partial^\alpha \widehat{\varphi}(\xi)| \lesssim (1 + |\xi|)^{-L}$ for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq N$. It is then straightforward to check that φ, ψ satisfy all assumptions of Propositions 8.3 and 8.4 (for proper choices of N, L, L_1, L_2). Hence, our general theory indeed yields *compactly supported* wavelet systems that form atomic decompositions and Banach frames for inhomogeneous Besov spaces.

- We observe that the assumptions of Propositions 8.3 and 8.4 are *structurally* very similar, but the precise values of L, L_1, L_2 differ greatly. Indeed, in order to get a *Banach frame* for $\mathcal{B}_s^{p,q}(\mathbb{R}^d)$ using Proposition 8.3, the mother wavelet ψ has to have at least $L_2 > s$ vanishing moments, which increases with the *smoothness parameter* $s \in \mathbb{R}$. In contrast, in order to obtain an *atomic decomposition* of $\mathcal{B}_s^{p,q}(\mathbb{R}^d)$ using Proposition 8.4, the mother wavelet ψ only has to have $L_2 > \vartheta_0 d - s$ vanishing moments, where $\vartheta_0 = (p^{-1} - 1)_+$. In particular, once the smoothness parameter s satisfies $s > d(p^{-1} - 1)_+$, one can choose $L_2 = 0$, so that it is possible for ψ to have *no vanishing moments at all*, i.e., $\widehat{\psi}(0) \neq 0$ is allowed.

In this case, one can even choose $\varphi = \psi$ to show that the system $\left(2^{j\frac{d}{2}} \cdot \varphi(2^j \bullet - \delta k) \right)_{j \in \mathbb{N}_0, k \in \mathbb{Z}^d}$ yields an atomic decompositions of $\mathcal{B}_s^{p,q}(\mathbb{R}^d)$. A peculiar property of this system is that it *does not even form a frame* for $L^2(\mathbb{R}^d)$, due to the missing vanishing moments.

- Wavelet characterizations of inhomogeneous Besov spaces have already been considered by many other authors: In [59, equations (10.1) and (10.2)], as well as in [74, Theorem 3.5(i)], it is shown that certain wavelet *orthonormal bases* yield atomic decompositions and Banach frames for the Besov spaces $\mathcal{B}_s^{p,q}(\mathbb{R}^d)$. We remark that of the two

mentioned books, only Triebel's book [74] covers the whole range $p, q \in (0, \infty]$, while Meyer[59] only considers the case $p, q \in [1, \infty]$.

As explained in [74, Theorem 1.61(ii)], the wavelet bases considered by Triebel in [74, Theorem 3.5] are *compactly supported* and are C^k , with k vanishing moments, where it is assumed that

$$k > \max \left\{ s, \frac{2d}{p} + \frac{d}{2} - s \right\}.$$

Hence, Triebel needs a large amount of vanishing moments if s is large, but also if $-s$ is large. As observed in the previous point, this is not needed for the theory developed in this paper, at least if one only wants to have *either* Banach frames or atomic decompositions. But since Triebel uses wavelet orthonormal bases, he obtains atomic decompositions and Banach frames *simultaneously*, which explains the dependence of k on s observed above.

In addition to orthonormal bases, Triebel also considers wavelet *frames*, cf. [74, Sections 1.8 and 3.2]. But for these, Triebel restricts to the case $p = q$. Then, for $s > \sigma_p = d(p^{-1} - 1)_+$, he derives atomic decomposition results using certain compactly supported wavelet frames (cf. [74, Theorem 1.69]). As seen above, this is the range in which Proposition 8.4 does not need any vanishing moments. Additional atomic decomposition results are obtained in [74, Theorem 1.71], but these use *bandlimited* wavelets and require $p > 1$ as well as $s < 0$.

In a different approach, Rauhut and Ullrich showed [69] (based upon previous work by Ullrich[75]) that the inhomogeneous Besov spaces $\mathcal{B}_s^{p,q}(\mathbb{R}^d)$ can be obtained as certain *generalized coorbit spaces*. Using the theory of these spaces (cf. [31, 69]), they then again show that suitable wavelet *orthonormal bases* yield Banach frames and atomic decompositions for the spaces $\mathcal{B}_s^{p,q}(\mathbb{R}^d)$, cf. [69, Theorem 5.8 and Remark 5.9]. Their assumptions on the scaling function φ and the mother wavelet ψ are very similar to the ones imposed in this paper: ψ needs to have a suitable number of vanishing moments, and φ, ψ are required to have a suitable decay in space, as well as in Fourier domain. Furthermore, the decay in Fourier domain also needs to hold for certain derivatives of $\widehat{\varphi}, \widehat{\psi}$, cf. [69, Definition 1.1]. We remark, however, that in [69, Theorem 5.8], only the range $p, q \in [1, \infty]$ is considered.

Finally, Frazier and Jawerth[34, 33, 32] also obtained atomic decompositions for Besov spaces, cf. [34, Theorem 7.1]. In contrast to our approach, Frazier and Jawerth use a sampling density which is fixed *a priori*. This, however, requires the mother wavelet ψ to be *bandlimited* to $\{\xi \in \mathbb{R}^d \mid |\xi| \leq \pi\}$ (cf. [34, between eq. (1.8) and eq. (1.9)]); in particular, ψ can *not* be compactly supported. We remark that Frazier and Jawerth assume ψ to have N vanishing moments with $N \geq \max \left\{ -1, d(p^{-1} - 1)_+ - s \right\}$. This is very similar to the vanishing moment condition which we impose in Proposition 8.4, cf. the preceding point. Finally, we mention the so-called **generalized φ -transform** of Frazier and Jawerth (cf. [33, Section 4]) which yields results that are very similar to Propositions 8.3 and 8.4, but for the case of (homogeneous) Triebel-Lizorkin spaces instead of inhomogeneous Besov spaces, cf. [32, Theorem 4.5] and [33, Corollaries 4.5 and 4.3]. For the case of *inhomogeneous* Triebel-Lizorkin spaces, see [33, Section 12, page 132].

In summary, we have seen that the description of (inhomogeneous) Besov spaces through wavelet systems—in particular through wavelet orthonormal bases—was very well developed prior to this paper. Nevertheless, it seems that in the case of compactly supported wavelet *frames* (as opposed to orthonormal bases), our results slightly improve the state of the art: In [74], comparable results are only derived for $p = q$ and $s > d(p^{-1} - 1)_+$ and in [34], only bandlimited wavelet systems are considered. Finally, in [33], the authors allow compactly supported wavelet frames, but consider Triebel-Lizorkin spaces instead of Besov spaces.

We close our comparison with the literature by comparing the advantages and disadvantages of wavelet orthonormal bases compared to more general wavelet systems. As noted in [44, Example 5.6(a)], “*both types of description are useful [...]: The orthogonal bases, when a concise characterization of a function without redundancy is important, but the form of the basic wavelet g is not essential; the non-orthogonal expansions and frames, when the basic function g is given by the problem and flexibility is required.*” Indeed, if one is willing to sample sufficiently densely, Propositions 8.3 and 8.4 allow a *very wide variety* of scaling functions φ and mother wavelets ψ to be used. In contrast, to obtain an orthonormal wavelet basis, φ and ψ need to be selected *very carefully*. However, using such an orthonormal basis has several advantages[74] that frames lack:

- the sampling density is known and fixed a priori,
- the synthesis coefficients are uniquely determined and equal to the analysis coefficients,
- the analysis map yields an isomorphism of $\mathcal{B}_s^{p,q}(\mathbb{R}^d)$ **onto** the associated sequence space $\ell_s^{p,q}$.
- Finally, we remark that we discussed inhomogeneous Besov spaces in the general framework presented here mainly to indicate that—and *how*—the framework can be applied in concrete cases. More novel and interesting applications of the general theory, in particular to shearlets, will be discussed in the companion paper [66]. ♦

APPENDIX A. LEMMAS NEEDED TO GET EXPLICIT CONSTANTS

Lemma A.1. *For each $N \in \mathbb{N}$, there is a polynomial $p_N \in \mathbb{R}[X]$ satisfying $0 \leq p_N(x) \leq 1$ for $x \in [0, 1]$ and*

$$p_N(0) = 0, \quad p_N(1) = 1, \quad \text{as well as} \quad p_N^{(\ell)}(0) = 0 = p_N^{(\ell)}(1) \quad \forall \ell \in \underline{N}.$$

Furthermore, p_N satisfies $\|p_N^{(\ell)}\|_{\sup, [0,1]} \leq 24^{N+1} \cdot (N+1)!$ for all $\ell \in \{0, \dots, N\}$. ◀

Proof. First, recall the well-known identity $\frac{d^\ell}{dx^\ell} x^N = \frac{N!}{(N-\ell)!} \cdot x^{N-\ell} = \binom{N}{\ell} \cdot \ell! \cdot x^{N-\ell}$ for $\ell \in \{0, \dots, N\}$. Now, define

$$q_N(x) := x^N \cdot (1-x)^N = x^N \cdot \sum_{m=0}^N \binom{N}{m} (-x)^m = \sum_{m=0}^N \left[\binom{N}{m} \cdot (-1)^m \cdot x^{N+m} \right]$$

and note for $x \in (0, 1]$ and $\ell \in \{0, \dots, N\}$ that

$$\begin{aligned} |q_N^{(\ell)}(x)| &\leq \sum_{m=0}^N \binom{N}{m} \binom{N+m}{\ell} \cdot \ell! \cdot x^{N+m-\ell} \\ &\stackrel{(\text{since } \binom{a}{b} \leq 2^a)}{\leq} \ell! \cdot x^{-\ell} \cdot \sum_{m=0}^N \left(\binom{N}{m} \cdot 2^{N+m} \cdot x^{N+m} \right) \\ &= \ell! \cdot x^{N-\ell} \cdot 2^N \cdot \sum_{m=0}^N \left(\binom{N}{m} \cdot (2x)^m \right) \\ &= \ell! \cdot x^{N-\ell} \cdot 2^N \cdot (1+2x)^N \\ &\stackrel{(\text{since } 0 < x \leq 1 \text{ and } N-\ell \geq 0)}{\leq} \ell! \cdot 2^N \cdot 3^N \leq 6^N \cdot N!. \end{aligned}$$

By continuity, this also holds for $x = 0$.

Furthermore, since we have $\frac{d^\ell}{dx^\ell} \Big|_{x=0} x^{N+m} = 0$ for all $\ell \leq N-1$ and all $m \geq 0$, we see $q_N^{(\ell)}(0) = 0$ for all $\ell \in \{0, \dots, N-1\}$. Likewise, note that

$$\begin{aligned} q_N(x) &= (-1)^N \cdot (x-1)^N \cdot (1+(x-1))^N \\ &= (-1)^N \cdot \sum_{m=0}^N \binom{N}{m} \cdot (x-1)^{N+m}, \end{aligned}$$

which implies $q_N^{(\ell)}(1) = 0$ for all $\ell \in \{0, \dots, N-1\}$, since $\frac{d^\ell}{dx^\ell} \Big|_{x=1} (x-1)^{N+m} = 0$ for all $\ell \in \{0, \dots, N-1\}$ and $m \geq 0$.

Next, note for $x \in [\frac{1}{2}(1 - \frac{1}{2N}), \frac{1}{2}(1 + \frac{1}{2N})] \subset [0, 1]$ that

$$x^N \geq 2^{-N} \cdot \left(1 - \frac{1}{2N}\right)^N = 2^{-N} \cdot \sqrt{\left(1 - \frac{1}{2N}\right)^{2N}} \geq 2^{-N} \cdot \sqrt{\left(1 - \frac{1}{2}\right)^2} = 2^{-(N+1)},$$

where we used the well-known fact that the sequence $[(1 - \frac{1}{n})^n]_{n \in \mathbb{N}}$ is nondecreasing⁴. Likewise, we get

$$(1-x)^N \geq \left[1 - \frac{1}{2} \left(1 + \frac{1}{2N}\right)\right]^N = \left[\frac{1}{2} \left(1 - \frac{1}{2N}\right)\right]^N \geq 2^{-(N+1)}$$

and thus $q_N(x) \geq 4^{-(N+1)}$, which yields

$$C_N := \int_0^1 q_N(t) dt \geq \int_{\frac{1}{2}(1-\frac{1}{2N})}^{\frac{1}{2}(1+\frac{1}{2N})} 4^{-(N+1)} dt = \frac{4^{-(N+1)}}{2N} = \frac{2^{-(2N+3)}}{N}.$$

Now, we finally define for $x \in [0, 1]$

$$p_N(x) := \frac{1}{C_N} \cdot \int_0^x q_N(t) dt = \frac{1}{C_N} \cdot \sum_{m=0}^N \left[\binom{N}{m} \cdot \frac{(-1)^m}{N+m+1} \cdot x^{N+m+1} \right]$$

⁴One way to see this is to note $\frac{d}{dx} (1 - \frac{1}{x})^x = (1 - \frac{1}{x})^x \cdot \left[\ln(1 - \frac{1}{x}) + \frac{1}{x-1} \right]$ as well as $\frac{d}{dx} \left[\ln(1 - \frac{1}{x}) + \frac{1}{x-1} \right] = \frac{1}{x-1} \left(\frac{1}{x} - \frac{1}{x-1} \right) < 0$ for $x \in (1, \infty)$ and $\ln(1 - \frac{1}{x}) + \frac{1}{x-1} \xrightarrow{x \rightarrow \infty} 0$. Together, these facts show $\frac{d}{dx} (1 - \frac{1}{x})^x > 0$ on $(1, \infty)$.

and note $p_N(0) = 0$, as well as $p_N(1) = \frac{1}{C_N} \cdot C_N = 1$, as desired. Also, the fundamental theorem of calculus shows

$$p_N^{(\ell)}(x) = \frac{1}{C_N} \cdot q_N^{(\ell-1)}(x) = 0 \quad \forall \ell \in \underline{N} \text{ and } x \in \{0, 1\}.$$

Furthermore, since $q_N \geq 0$, we see that p_N is nondecreasing and hence $0 = p_N(0) \leq p_N(x) \leq p_N(1) = 1$ for all $x \in [0, 1]$. Finally, we get

$$\left\| p_N^{(\ell)} \right\|_{\sup, [0, 1]} = \frac{1}{C_N} \cdot \left\| q_N^{(\ell-1)} \right\|_{\sup, [0, 1]} \leq 2^{2N+3} \cdot N \cdot 6^N \cdot N! \leq 24^N \cdot 8N \cdot N! \leq 24^{N+1} \cdot (N+1)!$$

for all $\ell \in \underline{N}$. For $\ell = 0$, this estimate is trivially satisfied since $\|p_N\|_{\sup, [0, 1]} = 1$. \square

Lemma A.2. *For all $d, N \in \mathbb{N}$ and $R, s > 0$ there is a function $\psi \in C_c^\infty(\mathbb{R}^d)$ satisfying*

- $0 \leq \psi \leq 1$,
- $\text{supp } \psi \subset (- (R + s), R + s)^d$,
- $\psi \equiv 1$ on $[-R, R]^d$,
- $\left\| \frac{\partial^\ell}{\partial x_i^\ell} \psi \right\|_{\sup} \leq \max \left\{ 1, \left(\frac{3}{s} \right)^\ell \right\} \cdot 24^{N+1} \cdot (N+1)!$ for all $i \in \underline{d}$ and all $\ell \in \{0, \dots, N\}$. \blacktriangleleft

Proof. Choose p_N as in Lemma A.1 and define

$$\psi^{(0)} : \mathbb{R} \rightarrow [0, 1], x \mapsto \begin{cases} 0, & \text{if } x \leq - (R + \frac{2}{3}s), \\ p_N \left(\frac{3}{s} \cdot (x + R + \frac{2}{3}s) \right), & \text{if } - (R + \frac{2}{3}s) \leq x \leq - (R + \frac{s}{3}), \\ 1, & \text{if } - (R + \frac{s}{3}) \leq x \leq R + \frac{s}{3}, \\ p_N \left(\frac{3}{s} \cdot [R + \frac{2}{3}s - x] \right), & \text{if } R + \frac{s}{3} \leq x \leq R + \frac{2}{3}s, \\ 0, & \text{if } x \geq R + \frac{2}{3}s. \end{cases}$$

Since we have $0 \leq p_N \leq 1$ and $p_N(0) = 0$, as well as $p_N(1) = 1$, it follows that $\psi^{(0)}$ is well-defined and continuous. Furthermore, it is well-known that if $f, g : \mathbb{R} \rightarrow \mathbb{R}$ are differentiable with $f'(a) = g'(a)$ and $f(a) = g(a)$, then so is

$$x \mapsto \begin{cases} f(x), & \text{if } x \leq a, \\ g(x), & \text{if } x \geq a. \end{cases}$$

By applying this inductively to higher derivatives and since $p_N^{(\ell)}(1) = 0 = p_N^{(\ell)}(0)$ for all $\ell \in \{1, \dots, N\}$, we conclude $\psi^{(0)} \in C^N(\mathbb{R})$, where the derivatives are obtained by differentiation of the individual “pieces” defining $\psi^{(0)}$. In particular, we get $\left\| \frac{d^\ell}{dx^\ell} \psi^{(0)} \right\|_{\sup} \leq \max \left\{ 1, \left(\frac{3}{s} \right)^\ell \right\} \cdot 24^{N+1} \cdot (N+1)!$ for all $\ell \in \{0, \dots, N\}$, cf. the estimate for the derivatives of p_N .

Now, let $\theta \in C_c^\infty((-1, 1))$ be a standard mollifier, i.e., $\theta \geq 0$ with $\int_{\mathbb{R}} \theta(t) dt = 1$. As usual, for $\varepsilon > 0$, let $\theta_\varepsilon(x) := \frac{1}{\varepsilon} \cdot \theta\left(\frac{x}{\varepsilon}\right)$ and $\psi_1 := \theta_{s/3} * \psi^{(0)}$. Using standard properties of convolution products, we see $\psi_1 \in C_c^\infty(\mathbb{R})$ with $0 \leq \psi_1 \leq 1$, as well as

$$\text{supp } \psi_1 \subset \left(-\frac{s}{3}, \frac{s}{3} \right) + \text{supp } \psi^{(0)} \subset (- (R + s), R + s)$$

and with

$$\left\| \frac{d^\ell}{dx^\ell} \psi_1 \right\|_{\sup} = \left\| \theta_{s/3} * \left[\frac{d^\ell}{dx^\ell} \psi^{(0)} \right] \right\|_{\sup} \leq \left\| \frac{d^\ell}{dx^\ell} \psi^{(0)} \right\|_{\sup} \leq \max \left\{ 1, \left(\frac{3}{s} \right)^\ell \right\} \cdot 24^{N+1} \cdot (N+1)!$$

for all $\ell \in \{0, \dots, N\}$.

Finally, for $x \in [-R, R]$, we have

$$\begin{aligned} \psi_1(x) &= \int_{-\infty}^{\infty} \theta_{s/3}(y) \cdot \psi^{(0)}(x-y) dy \\ &= \int_{-s/3}^{s/3} \theta_{s/3}(y) \cdot \psi^{(0)}(x-y) dy \\ &= \int_{-s/3}^{s/3} \theta_{s/3}(y) dy \quad (\psi^{(0)}(x-y) = 1 \text{ since } x-y \in [- (R + \frac{s}{3}), R + \frac{s}{3}]) \\ &= \int_{-\infty}^{\infty} \theta_{s/3}(y) dy = 1, \end{aligned}$$

as desired.

The preceding considerations establish the claim for $d = 1$. In case of $d > 1$, set $\psi := \psi_1 \otimes \cdots \otimes \psi_1$ and note

$$\frac{\partial^\ell}{\partial x_i^\ell} \psi = \bigotimes_{j=1}^{i-1} \psi_1 \otimes \psi_1^{(\ell)} \otimes \bigotimes_{j=i+1}^d \psi_1,$$

which yields the desired estimate for the derivative, since $0 \leq \psi_1 \leq 1$. \square

Corollary A.3. *The function ψ from Lemma A.2 satisfies*

$$|(\partial^\alpha [\mathcal{F}^{-1}\psi])(x)| \leq 2\pi \cdot 2^d \cdot \max\left\{1, (3/s)^N\right\} \cdot (48d)^{N+1} (N+2)! \cdot (1+R+s)^{|\alpha|} (R+s)^d \cdot (1+|x|)^{-N},$$

for all $x \in \mathbb{R}^d$ and all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq 1$.

In particular, we have for $N > d$ that

$$\|\nabla [\mathcal{F}^{-1}\psi]\|_{L^1} \leq \frac{2\pi \cdot s_d}{N-d} \cdot \sqrt{d} \cdot 2^d \cdot \max\left\{1, (3/s)^N\right\} \cdot (48d)^{N+1} (N+2)! \cdot (1+R+s) (R+s)^d. \quad \blacktriangleleft$$

Proof. We first recall the elementary identity

$$\begin{aligned} (\partial^\alpha [\mathcal{F}^{-1}\psi])(x) &= \int_{\mathbb{R}^d} \psi(\xi) \cdot \partial_x^\alpha e^{2\pi i \langle x, \xi \rangle} d\xi \\ &= \int_{\mathbb{R}^d} \psi(\xi) \cdot (2\pi i \xi)^\alpha \cdot e^{2\pi i \langle x, \xi \rangle} d\xi \\ &= (\mathcal{F}^{-1}[\xi \mapsto (2\pi i \xi)^\alpha \cdot \psi(\xi)])(x). \end{aligned}$$

Hence, we let $g : \mathbb{R}^d \rightarrow \mathbb{C}$, $\xi \mapsto (2\pi i \xi)^\alpha \cdot \psi(\xi)$, note $g \in C_c^\infty(\mathbb{R}^d)$ and recall from Lemma 6.3, equation (6.2) that

$$|(\mathcal{F}^{-1}g)(x)| \leq (1+|x|)^{-N} \cdot (1+d)^N \cdot \left(|(\mathcal{F}^{-1}g)(x)| + \sum_{m=1}^d |[\mathcal{F}^{-1}(\partial_m^N g)](x)| \right) \quad (\text{A.1})$$

for all $x \in \mathbb{R}^d$. Thus, it remains to estimate the right-hand side.

But for the first term, we simply have because of $\text{supp } g \subset \text{supp } \psi \subset (-(R+s), R+s)^d$ and $0 \leq \psi \leq 1$, which entails $|g(\xi)| \leq [2\pi(R+s)]^{|\alpha|}$ for all $\xi \in \mathbb{R}^d$, that

$$|(\mathcal{F}^{-1}g)(x)| \leq \|g\|_{L^1} \leq [2\pi(R+s)]^{|\alpha|} \cdot [2(R+s)]^d \leq 2\pi \cdot 2^d \cdot (1+R+s)^{|\alpha|} (R+s)^d.$$

For the second term, we have to work harder: In case of $\alpha = 0$, we simply have $g = \psi$ and hence—as above—that

$$\begin{aligned} |[\mathcal{F}^{-1}(\partial_m^N g)](x)| &\leq \|\partial_m^N g\|_{L^1} \leq [2(R+s)]^d \cdot \|\partial_m^N g\|_{L^\infty} \\ &\leq \max\left\{1, \left(\frac{3}{s}\right)^N\right\} \cdot 24^{N+1} \cdot (N+1)! \cdot [2(R+s)]^d \\ &\leq 2\pi \cdot 2^d \cdot \max\left\{1, \left(\frac{3}{s}\right)^N\right\} \cdot 24^{N+1} \cdot (N+2)! \cdot (R+s)^d (1+R+s)^{|\alpha|}, \end{aligned}$$

cf. Lemma A.2 for the estimate regarding $\|\partial_m^N g\|_{L^\infty} = \|\partial_m^N \psi\|_{L^\infty}$.

It remains to consider the case $|\alpha| = 1$, i.e., $\alpha = e_j$ for some $j \in \underline{d}$. In this case, we have $g(\xi) = 2\pi i \cdot \xi_j \cdot \psi(\xi)$, so that Leibniz's rule yields

$$\begin{aligned} |(\partial_m^N g)(\xi)| &= 2\pi \cdot \left| \sum_{\ell=0}^N \binom{N}{\ell} \cdot [\partial_m^\ell \xi_j] \cdot (\partial_m^{N-\ell} \psi)(\xi) \right| \\ &\quad (\text{since } |\xi_j| \leq R+s \text{ on } \text{supp } \psi \text{ and } \partial_m^\ell \xi_j = \delta_{m,j} \cdot \delta_{\ell,1} \text{ for } \ell \geq 1) \leq 2\pi \cdot [(R+s) \cdot |(\partial_m^N \psi)(\xi)| + N \cdot |(\partial_m^{N-1} \psi)(\xi)|] \\ &\quad (\text{cf. Lemma A.2}) \leq 2\pi N \cdot (1+R+s) \cdot \max\left\{1, \left(\frac{3}{s}\right)^N\right\} \cdot 24^{N+1} \cdot (N+1)! \\ &\quad (\text{since } |\alpha|=1) \leq 2\pi \cdot (1+R+s)^{|\alpha|} \cdot \max\left\{1, \left(\frac{3}{s}\right)^N\right\} \cdot 24^{N+1} \cdot (N+2)!. \end{aligned}$$

Here, we used that $\max \left\{ 1, (3/s)^\ell \right\}$ is nondecreasing with respect to $\ell \in \{0, \dots, N\}$. In fact, for $3/s \leq 1$, we have $\max \left\{ 1, (3/s)^\ell \right\} = 1$ for all $\ell \in \{0, \dots, N\}$ and for $3/s > 1$, we have $\max \left\{ 1, (3/s)^\ell \right\} = (3/s)^\ell$, which is increasing with respect to ℓ . All in all, we arrive at

$$\begin{aligned} |[\mathcal{F}^{-1}(\partial_m^N g)](x)| &\leq \|\partial_m^N g\|_{L^1} \leq [2(R+s)]^d \cdot \|\partial_m^N g\|_{L^\infty} \\ &\leq 2\pi \cdot \max \left\{ 1, \left(\frac{3}{s} \right)^N \right\} \cdot 24^{N+1} \cdot (N+2)! \cdot [2(R+s)]^d \cdot (1+R+s)^{|\alpha|} \\ &\leq 2\pi \cdot 2^d \cdot \max \left\{ 1, \left(\frac{3}{s} \right)^N \right\} \cdot 24^{N+1} \cdot (N+2)! \cdot (R+s)^d \cdot (1+R+s)^{|\alpha|}. \end{aligned}$$

Recalling equation (A.1), we conclude

$$\begin{aligned} (1+|x|)^N \cdot |(\mathcal{F}^{-1}g)(x)| &\leq (1+d)^N \left(2\pi \cdot 2^d (1+R+s)^{|\alpha|} (R+s)^d + d \cdot 2\pi \cdot 2^d \max \left\{ 1, \left(\frac{3}{s} \right)^N \right\} \cdot 24^{N+1} (N+2)! \cdot (R+s)^d (1+R+s)^{|\alpha|} \right) \\ &\leq 2\pi \cdot 2^d (1+d)^{N+1} \cdot \max \left\{ 1, \left(\frac{3}{s} \right)^N \right\} \cdot 24^{N+1} (N+2)! \cdot (1+R+s)^{|\alpha|} (R+s)^d \\ &\leq 2\pi \cdot 2^d \cdot \max \left\{ 1, \left(\frac{3}{s} \right)^N \right\} \cdot (48d)^{N+1} (N+2)! \cdot (1+R+s)^{|\alpha|} (R+s)^d, \end{aligned}$$

as claimed.

For the additional claim, recall from equation (1.9) for $N > d$ that

$$\begin{aligned} \|\nabla [\mathcal{F}^{-1}\psi]\|_{L^1} &\leq 2\pi\sqrt{d} \cdot 2^d \cdot \max \left\{ 1, \left(\frac{3}{s} \right)^N \right\} \cdot (48d)^{N+1} (N+2)! \cdot (1+R+s) (R+s)^d \cdot \left\| (1+|\cdot|)^{-N} \right\|_{L^1} \\ &\leq \frac{2\pi \cdot s_d}{N-d} \cdot \sqrt{d} \cdot 2^d \cdot \max \left\{ 1, \left(\frac{3}{s} \right)^N \right\} \cdot (48d)^{N+1} (N+2)! \cdot (1+R+s) (R+s)^d, \end{aligned}$$

where the first step is justified by a combination of our previous estimates with the Cauchy-Schwarz inequality. \square

APPENDIX B. VANISHING OF A FUNCTION IMPLIES VANISHING OF DERIVATIVES

In this section, we show that if a sufficiently smooth function $f : U \subset \mathbb{R}^d \rightarrow \mathbb{R}$ satisfies $f(x) \in o(|x-a|^N)$ as $x \rightarrow a$, then the partial derivatives of f also vanish to a suitable order at a , i.e., $\partial^\alpha f(x) \in o(|x-a|^{N-|\alpha|})$ as $x \rightarrow a$, for $|\alpha| \leq N$. Our starting point is the following consequence of Taylor's theorem:

Lemma B.1. *Let $a \in \mathbb{R}^d$, $r > 0$ and $N \in \mathbb{N}_0$. Assume that $f \in C^N(B_r(a); \mathbb{R})$ satisfies $f(x) \in o(|x-a|^N)$ as $x \rightarrow a$, i.e.,*

$$\frac{f(x)}{|x-a|^N} \xrightarrow{x \rightarrow a} 0.$$

Then $\partial^\alpha f(a) = 0$ for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq N$. \blacktriangleleft

Proof. Let f as in the statement of the lemma. We will show by induction on $\ell = |\alpha| \in \{0, \dots, N\}$ that $\partial^\alpha f(a) = 0$. By translating everything, we can clearly assume $a = 0$.

For $|\alpha| = 0$, we simply note by continuity of f at $a = 0$ that

$$f(0) = \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{f(x)}{|x|^N} \cdot \lim_{x \rightarrow 0} |x|^N = 0,$$

since $\lim_{x \rightarrow 0} |x|^N \in \{0, 1\}$ because of $N \in \mathbb{N}_0$.

Now, assume $\partial^\alpha f(0) = 0$ for all $|\alpha| < \ell$ for some $\ell \in \{1, \dots, N\}$. In view of Taylor's theorem (cf. [3, Theorem 5.11] for the precise version used here), we get

$$f(x) = \sum_{|\alpha| \leq N} \frac{\partial^\alpha f(0)}{\alpha!} x^\alpha + R_N(x) \quad \forall x \in B_r(0),$$

where $R_N : B_r(0) \rightarrow \mathbb{R}$ satisfies $R_N(x) \in o(|x|^N)$, i.e., $R_N(x) / |x|^N \xrightarrow{x \rightarrow 0} 0$. By rearranging, and since $\partial^\alpha f(0) = 0$ for all $|\alpha| < \ell$, we get

$$p(x) := \sum_{|\alpha|=\ell} \frac{\partial^\alpha f(0)}{\alpha!} x^\alpha = f(x) - \sum_{\ell+1 \leq |\alpha| \leq N} \frac{\partial^\alpha f(0)}{\alpha!} x^\alpha - R_N(x) =: g(x) \quad \forall x \in B_r(0).$$

But we have $R_N(x) \in o(|x|^N) \subset o(|x|^\ell)$ and likewise $f(x) \in o(|x|^N) \subset o(|x|^\ell)$ as $x \rightarrow 0$. Also, $|x^\alpha| \leq |x|^{|\alpha|} \leq |x|^{\ell+1}$ for $|x| < 1$ and $\ell+1 \leq |\alpha| \leq N$, so that $\frac{|x^\alpha|}{|x|^\ell} \leq |x| \xrightarrow{x \rightarrow 0} 0$. All in all, we thus see $g(x) \in o(|x|^\ell)$ as $x \rightarrow 0$ and hence also $p(x) \in o(|x|^\ell)$.

Next, if we can show $p \equiv 0$, it follows from standard properties of polynomials that $\frac{\partial^\alpha f(0)}{\alpha!} = 0$ for all $|\alpha| = \ell$ and hence $\partial^\alpha f(0) = 0$ for all $|\alpha| = \ell$. One possibility of proving this elementary fact is to note that

$$\partial^\beta x^\alpha = \begin{cases} 0, & \text{if } \alpha \neq \beta, \\ c_\alpha > 0, & \text{if } \alpha = \beta, \end{cases}$$

for $\alpha, \beta \in \mathbb{N}_0^d$ with $|\alpha| = |\beta|$.

Thus, all we need to show is that if $p(x) = \sum_{|\alpha|=\ell} c_\alpha x^\alpha$ satisfies $p \in o(|x|^\ell)$ as $x \rightarrow 0$ then $p \equiv 0$. It is clear that $p(0) = 0$, since $\ell \geq 1$. Now let $x \in \mathbb{R}^d \setminus \{0\}$ be arbitrary. Since p is homogeneous of degree ℓ , we have $p(rx) = r^\ell \cdot p(x)$ for all $r > 0$. Using this and $p \in o(|x|^\ell)$, we get

$$0 = \lim_{r \downarrow 0} \frac{p(rx)}{|rx|^\ell} = \lim_{r \downarrow 0} \frac{p(x)}{|x|^\ell} = \frac{p(x)}{|x|^\ell}$$

and hence $p(x) = 0$ for all $x \in \mathbb{R}^d$, as desired. \square

Lemma B.2. *Let $U \subset \mathbb{R}^d$ be open, let $N \in \mathbb{N}_0$ and $f \in C^N(U; \mathbb{R})$. If f satisfies $f(x) / |x - a|^N \rightarrow 0$ as $x \rightarrow a$ for some $a \in U$, then*

$$\frac{\partial^\alpha f(x)}{|x - a|^{N-|\alpha|}} \xrightarrow{x \rightarrow a} 0 \quad \forall \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \leq N. \quad \blacktriangleleft$$

Remark. The lemma remains true for complex-valued functions, since one can simply apply it to the real- and imaginary parts separately. \blacklozenge

Proof. Lemma B.1 shows $\partial^\gamma f(a) = 0$ for all $\gamma \in \mathbb{N}_0^d$ with $|\gamma| \leq N$. Hence, for $|\alpha| = N$ we get by continuity of $\partial^\alpha f$ that

$$\frac{\partial^\alpha f(x)}{|x - a|^{N-|\alpha|}} = \partial^\alpha f(x) \xrightarrow{x \rightarrow a} \partial^\alpha f(a) = 0,$$

as desired. Hence, we can assume $|\alpha| < N$ from now on. This implies $g := \partial^\alpha f \in C^{N-|\alpha|}(U; \mathbb{R})$, so that Taylor's theorem (see [3, Theorem 5.11] for the precise version used here) yields because of $\partial^\beta g(a) = \partial^{\alpha+\beta} f(a) = 0$ for $|\beta| \leq N - |\alpha|$ that

$$g(x) = \sum_{|\beta| \leq N-|\alpha|} \frac{\partial^\beta g(a)}{\beta!} (x-a)^\beta + o(|x-a|^{N-|\alpha|}) = o(|x-a|^{N-|\alpha|}) \quad \text{as } x \rightarrow a,$$

as claimed. \square

APPENDIX C. NECESSITY OF VANISHING MOMENT CONDITIONS FOR DISCRETE CONE-ADAPTED SHEARLET FRAMES

Proposition C.1. *Let $\varphi, \gamma, \tilde{\gamma} \in L^2(\mathbb{R}^2)$ such that for some $\delta > 0$, the (discrete, cone-adapted) shearlet system (cf. [54, Definition 2.2]) with sampling density δ ,*

$$\mathcal{SH}(\varphi, \gamma, \tilde{\gamma}; \delta) = \Phi(\varphi; \delta) \cup \Psi(\gamma; \delta) \cup \tilde{\Psi}(\tilde{\gamma}; \delta)$$

with

$$\begin{aligned} \Phi(\varphi; \delta) &= \{ \phi(\bullet - \delta m) \mid m \in \mathbb{Z}^2 \}, \\ \Psi(\gamma; \delta) &= \left\{ \gamma_{j,k,m} = 2^{\frac{3}{4}j} \cdot \gamma(S_k A_{2^j} \bullet - \delta m) \mid (j,k) \in I \text{ and } m \in \mathbb{Z}^2 \right\} \\ \tilde{\Psi}(\tilde{\gamma}; \delta) &= \left\{ \tilde{\gamma}_{j,k,m} = 2^{\frac{3}{4}j} \cdot \tilde{\gamma}(S_k^T A_{2^j} \bullet - \delta m) \mid (j,k) \in I \text{ and } m \in \mathbb{Z}^2 \right\} \end{aligned}$$

and $I := \{(j, k) \in \mathbb{N}_0 \times \mathbb{Z} : |k| \leq \lceil 2^{j/2} \rceil\}$, as well as

$$S_k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}, \quad A_a = \text{diag}(a, \sqrt{a}), \quad \text{and} \quad \tilde{A}_a = \text{diag}(\sqrt{a}, a)$$

is a **Bessel system** in $L^2(\mathbb{R}^2)$. Then we have

$$\int_{\{\xi \in \mathbb{R}^2 : |\xi_1| < 1 \text{ and } |\xi_2| < \frac{1}{8}|\xi_1|^{1/2}\}} |\xi_1|^{-2} \cdot |\widehat{\gamma}(\xi)|^2 d\xi < \infty. \quad \blacktriangleleft$$

Remark. • Here, a system $(\theta_i)_{i \in I}$ in a Hilbert space \mathcal{H} is called a Bessel system if there is a constant $C > 0$ satisfying $\sum_{i \in I} |\langle f, \theta_i \rangle_{\mathcal{H}}|^2 \leq C \cdot \|f\|_{\mathcal{H}}^2$ for all $f \in \mathcal{H}$.

- Likewise, one can show

$$\int_{\{\xi \in \mathbb{R}^2 : |\xi_2| < 1 \text{ and } |\xi_1| < \frac{1}{8}|\xi_2|^{1/2}\}} |\xi_2|^{-2} \cdot |(\mathcal{F}\widehat{\gamma})(\xi)|^2 d\xi < \infty.$$

- In particular, if $\widehat{\gamma}$ is continuous (e.g. if γ is compactly supported), then necessarily $\widehat{\gamma}(0) = 0$, since otherwise $|\widehat{\gamma}(\xi)| \geq c$ for $|\xi| < 2\varepsilon$ with $c > 0$ and $\varepsilon \in (0, 1)$ suitable. But this yields

$$\begin{aligned} \int_{\{\xi \in \mathbb{R}^2 : |\xi_1| < 1 \text{ and } |\xi_2| < \frac{1}{8}|\xi_1|^{1/2}\}} |\xi_1|^{-2} \cdot |\widehat{\gamma}(\xi)|^2 d\xi &\geq c^2 \cdot \int_{\{\xi \in \mathbb{R}^2 : |\xi_1| < \varepsilon^2 \text{ and } |\xi_2| < \frac{1}{8}|\xi_1|^{1/2}\}} |\xi_1|^{-2} d\xi \\ &\stackrel{(\text{Fubini's theorem})}{\geq} c^2 \cdot \int_0^{\varepsilon^2} |\xi_1|^{-2} \cdot \frac{1}{4} |\xi_1|^{1/2} d\xi_1 \\ &= \frac{c^2}{4} \cdot \int_0^{\varepsilon^2} |\xi_1|^{-\frac{3}{2}} d\xi_1 = \infty. \quad \blacklozenge \end{aligned}$$

Proof. The following proof is heavily inspired by the proof of [19, Theorem 3.3.1], generalized from wavelets to shearlets and from homogeneous systems to inhomogeneous systems.

In the following, we will consider the **shearlet group**

$$H = \left\{ \varepsilon \begin{pmatrix} a & b \\ 0 & \sqrt{a} \end{pmatrix} \mid a > 0, b \in \mathbb{R}, \varepsilon \in \{\pm 1\} \right\} = \left\{ \begin{pmatrix} a & b \\ 0 & \text{sgn}(a) \cdot \sqrt{|a|} \end{pmatrix} \mid a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R} \right\},$$

which contains all of the matrices S_k and A_a . We let μ_H denote the **Haar measure** (cf. [28, Section 2.2]) on the locally compact topological group H . Based on μ_H , we define a new measure ν on the Borel σ -algebra of $\mathbb{R}^2 \times H$ by

$$\nu(A) = \int_H \int_{\mathbb{R}^2} \frac{\mathbb{1}_A(x, h)}{|\det h|} dx d\mu_H(h) \quad \text{for each Borel set } A \subset \mathbb{R}^2 \times H.$$

For $f, \psi \in L^2(\mathbb{R}^2)$, the **continuous shearlet transform** $W_\psi f$ is given by

$$W_\psi f : \mathbb{R}^2 \times H \rightarrow \mathbb{C}, (x, h) \mapsto \langle f, \pi(x, h)\psi \rangle,$$

where $\pi(x, h)\psi := L_x D_h \psi$, with $L_x f(y) = f(y - x)$ and $D_h f = |\det h|^{-1/2} \cdot f \circ h^{-1}$. It is not hard to show that the inverse of the operator $\pi(x, h) = L_x D_h$ is given by $[\pi(x, h)]^{-1} = \pi(-h^{-1}x, h^{-1})$ and furthermore that $\pi(x, h)\pi(y, g) = \pi(x + hy, hg)$ for arbitrary $x, y \in \mathbb{R}^2$ and $g, h \in H$.

Since we have $|W_\psi f(x, h)| \leq \|f\|_{L^2} \cdot \|\psi\|_{L^2}$ and since $\|\pi(x, h)\psi\|_{L^2} = \|\psi\|_{L^2}$ for all $(x, h) \in \mathbb{R}^2 \times H$, it is not hard to see for $F \in L^1(\nu; \mathbb{C})$ that

$$T_F f := \int_{\mathbb{R}^2 \times H} F(x, h) \cdot W_\psi f(x, h) \cdot \pi(x, h)\psi d\nu(x, h) \in L^2(\mathbb{R}^2)$$

is well-defined with $\|T_F f\|_{L^2} \leq \|F\|_{L^1(\nu)} \cdot \|\psi\|_{L^2}^2 \cdot \|f\|_{L^2}$. Furthermore, in case of $F \geq 0$, we have

$$\begin{aligned} \langle T_F f, f \rangle_{L^2} &= \int_{\mathbb{R}^2 \times H} F(x, h) \cdot W_\psi f(x, h) \cdot \langle \pi(x, h)\psi, f \rangle_{L^2} d\nu(x, h) \\ &\stackrel{(\text{since } \langle \pi(x, h)\psi, f \rangle_{L^2} = \overline{\langle f, \pi(x, h)\psi \rangle_{L^2}} = \overline{W_\psi f(x, h)})}{=} \int_{\mathbb{R}^2 \times H} F(x, h) \cdot |W_\psi f(x, h)|^2 d\nu(x, h) \geq 0, \end{aligned}$$

so that the operator $T_F : L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)$ is bounded and nonnegative and in particular self-adjoint.

Finally, if $F \in L^1(\nu)$ is nonnegative and if $(u_\ell)_\ell$ is an arbitrary orthonormal basis of $L^2(\mathbb{R}^2)$, then

$$\sum_\ell \langle T_F u_\ell, u_\ell \rangle_{L^2} = \int_{\mathbb{R}^2 \times H} F(x, h) \cdot \sum_\ell |W_\psi u_\ell(x, h)|^2 d\nu(x, h)$$

$$(\text{since } \sum_\ell |W_\psi u_\ell(x, h)|^2 = \sum_\ell |\langle u_\ell, \pi(x, h)\psi \rangle|^2 = \|\pi(x, h)\psi\|_{L^2}^2 = \|\psi\|_{L^2}^2) = \|\psi\|_{L^2}^2 \cdot \|F\|_{L^1(\nu)} < \infty.$$

Thus, T_F is a **trace-class operator** (cf. [28, Appendix 2]) and in particular a compact operator.

Hence, if $F \in L^1(\nu)$ and $F \geq 0$, then the spectral theorem for compact self-adjoint operators yields an orthonormal basis $(u_\ell)_{\ell \in \mathbb{N}}$ of $L^2(\mathbb{R}^2)$ satisfying $T_F = \sum_{\ell=1}^\infty c_\ell \cdot \langle \bullet, u_\ell \rangle \cdot u_\ell$ where $c_\ell \geq 0$ and $\sum_{\ell=1}^\infty c_\ell = \|\psi\|_{L^2}^2 \cdot \|F\|_{L^1(\nu)} < \infty$.

Now, if $(g_i)_{i \in I}$ is an arbitrary Bessel-sequence in $L^2(\mathbb{R}^2)$, i.e., if $\sum_{i \in I} |\langle f, g_i \rangle|^2 \leq C \cdot \|f\|_{L^2}^2$ for each $f \in L^2(\mathbb{R}^2)$, then

$$\begin{aligned} \sum_{i \in I} \langle T_F g_i, g_i \rangle &= \sum_{i \in I} \sum_{\ell=1}^\infty c_\ell \cdot \langle g_i, u_\ell \rangle \langle u_\ell, g_i \rangle \\ &= \sum_{\ell=1}^\infty c_\ell \sum_{i \in I} |\langle u_\ell, g_i \rangle|^2 \\ &(\text{since } \|u_\ell\|_{L^2}=1) \leq C \cdot \sum_{\ell=1}^\infty c_\ell = C \cdot \|F\|_{L^1(\nu)} \cdot \|\psi\|_{L^2}^2 < \infty. \end{aligned} \tag{C.1}$$

Next, choose an arbitrary compact set $\Lambda \subset H_+ := \left\{ \begin{pmatrix} a & b \\ 0 & \sqrt{a} \end{pmatrix} \mid a \in (0, \infty) \text{ and } b \in \mathbb{R} \right\}$ with nonempty interior. By compactness, the constant $C_0 := \sup_{\lambda \in \Lambda} \|\lambda^{-1}\|_{\ell^\infty \rightarrow \ell^\infty}$ is finite. Now, define $\omega := \mathbb{1}_{[-3\delta C_0, 3\delta C_0]^2} \in L^1(\mathbb{R}^2)$ and let

$$F : \mathbb{R}^2 \times H \rightarrow [0, \infty), (x, h) \mapsto \omega(h^{-1}x) \cdot \mathbb{1}_\Lambda(h).$$

With this definition, we have

$$\begin{aligned} \|F\|_{L^1(\nu)} &= \int_\Lambda \int_{\mathbb{R}^2} |\det h|^{-1} \cdot \omega(h^{-1}x) dx d\mu_H(h) \\ (\text{for } y=h^{-1}x) &= \int_\Lambda \int_{\mathbb{R}^2} \omega(y) dy d\mu_H(h) = (6\delta C_0)^2 \cdot \mu_H(\Lambda) < \infty. \end{aligned}$$

Next, we define

$$Q_0 := \left\{ \xi \in \mathbb{R}^* \times \mathbb{R} \mid |\xi_1| \in [2^{-1}, 1) \text{ and } \frac{\xi_2}{\xi_1} \in [-2^{-1}, 2^{-1}) \right\}$$

and set $Q := \Lambda^T \overline{Q_0}$, where it is not hard to see that $Q \subset \mathbb{R}^* \times \mathbb{R}$ is compact. Consequently, there is some $\psi \in \mathcal{S}(\mathbb{R}^2) \subset L^2(\mathbb{R}^2)$ with $\hat{\psi} \in C_c^\infty(\mathbb{R}^* \times \mathbb{R})$ and $\hat{\psi} \geq 0$, as well as $\hat{\psi} \equiv 1$ on Q .

Now, we observe that the part $\Psi(\gamma; \delta) = \{\gamma_{j,k,m} \mid (j,k) \in I \text{ and } m \in \mathbb{Z}^2\}$ of the shearlet system $\mathcal{SH}(\varphi, \gamma, \tilde{\gamma}; \delta)$ satisfies

$$\begin{aligned} \gamma_{j,k,m} &= 2^{\frac{3}{4}j} \cdot \gamma(S_k A_{2^j} \bullet - \delta m) \\ &= 2^{\frac{3}{4}j} \cdot \gamma\left(S_k A_{2^j} \left[\bullet - (S_k A_{2^j})^{-1} \delta m \right]\right) \\ &= L_{x_{j,k,m}} D_{(S_k A_{2^j})^{-1}} \gamma = \pi\left(x_{j,k,m}, (S_k A_{2^j})^{-1}\right) \gamma \quad \text{with} \quad x_{j,k,m} := (S_k A_{2^j})^{-1} \delta m \in \mathbb{R}^2 \end{aligned}$$

for $(j,k) \in I$ and $m \in \mathbb{Z}^2$. Consequently, since each map $\pi(x, h)$ is unitary,

$$\begin{aligned} [W_\psi \gamma_{j,k,m}](x, h) &= \langle \gamma_{j,k,m}, \pi(x, h)\psi \rangle = \left\langle \pi\left(x_{j,k,m}, (S_k A_{2^j})^{-1}\right) \gamma, \pi(x, h)\psi \right\rangle \\ &= \langle \gamma, \pi(-S_k A_{2^j} x_{j,k,m}, S_k A_{2^j}) \pi(x, h)\psi \rangle \\ &= \langle \gamma, \pi(S_k A_{2^j} x - \delta m, S_k A_{2^j} \cdot h)\psi \rangle \quad \forall (x, h) \in \mathbb{R}^2 \times H. \end{aligned}$$

Since $\mathcal{SH}(\varphi, \gamma, \tilde{\gamma}; \delta)$ is a Bessel system, so is $\Psi(\gamma; \delta)$. In view of equation (C.1), this implies

$$\sum_{(j,k) \in I, m \in \mathbb{Z}^2} \langle T_F \gamma_{j,k,m}, \gamma_{j,k,m} \rangle < \infty.$$

Now, let $(j, k) \in I$ and $m \in \mathbb{Z}^2$ be arbitrary and note

$$\begin{aligned}
\langle T_F \gamma_{j,k,m}, \gamma_{j,k,m} \rangle &= \int_{\mathbb{R}^2 \times H} F(x, h) \cdot |W_\psi \gamma_{j,k,m}(x, h)|^2 d\nu(x, h) \\
&= \int_H \mathbf{1}_\Lambda(h) \cdot \int_{\mathbb{R}^2} |\det h|^{-1} \cdot \omega(h^{-1}x) \cdot |\langle \gamma, \pi(S_k A_{2^j} x - \delta m, S_k A_{2^j} \cdot h) \psi \rangle|^2 dx d\mu_H(h) \\
&\quad (\text{for } z = h^{-1}x) = \int_H \mathbf{1}_\Lambda(h) \cdot \int_{\mathbb{R}^2} \omega(z) \cdot |\langle \gamma, \pi(S_k A_{2^j} \cdot h z - \delta m, S_k A_{2^j} \cdot h) \psi \rangle|^2 dz d\mu_H(h) \\
&\quad (\text{for } g = S_k A_{2^j} h) = \int_H \mathbf{1}_\Lambda((S_k A_{2^j})^{-1} g) \cdot \int_{\mathbb{R}^2} \omega(z) \cdot |\langle \gamma, \pi(g \cdot z - \delta m, g) \psi \rangle|^2 dz d\mu_H(g) \\
&\quad (\text{for } x = g \cdot z - \delta m) = \int_H \mathbf{1}_{S_k A_{2^j} \Lambda}(g) \cdot \int_{\mathbb{R}^2} \omega(g^{-1}[x + \delta m]) \cdot |\langle \gamma, \pi(x, g) \psi \rangle|^2 dx \frac{d\mu_H(g)}{|\det g|}. \tag{C.2}
\end{aligned}$$

Next, let $(j, k) \in I$ and $x \in \mathbb{R}^2$ be fixed and let $g \in S_k A_{2^j} \Lambda$, i.e., $g = S_k A_{2^j} \lambda$ for some $\lambda \in \Lambda$. Observe that $\mathbb{Z}^2 \rightarrow \mathbb{Z}^2, m \mapsto S_k m$ is a bijection, so that

$$\begin{aligned}
\sum_{m \in \mathbb{Z}^2} \omega(g^{-1}[x + \delta m]) &= \sum_{m \in \mathbb{Z}^2} \omega(\lambda^{-1} A_{2^j}^{-1} [S_k^{-1} x + \delta S_k^{-1} m]) \\
&\quad (\text{with } n = S_k^{-1} m) = \sum_{n \in \mathbb{Z}^2} \omega(\lambda^{-1} A_{2^j}^{-1} [S_k^{-1} x + \delta n]). \tag{C.3}
\end{aligned}$$

But we have $\mathbb{R}^2 = \bigsqcup_{n \in \mathbb{Z}^2} [n + [0, 1]^2]$, so that there is some $n_0 = n_0(x, k, \delta) \in \mathbb{Z}^2$ satisfying $-\frac{1}{\delta} S_k^{-1} x \in n_0 + [0, 1]^2$. Hence, $\|S_k^{-1} x + \delta n_0\|_\infty = \delta \left\| -\frac{1}{\delta} S_k^{-1} x - n_0 \right\|_\infty \leq \delta$. Now, let

$$Z_j := \{-2^j, \dots, 2^j\} \times \{-\lceil 2^{j/2} \rceil, \dots, \lceil 2^{j/2} \rceil\}$$

and observe for $n \in n_0 + Z_j$ that

$$\begin{aligned}
S_k^{-1} x + \delta n &= S_k^{-1} x + \delta n_0 + \delta(n - n_0) \in \delta \left([-1, 1]^2 + Z_j \right) \\
&\subset \delta \cdot \left([-(1 + 2^j), 1 + 2^j] \times [-(1 + \lceil 2^{j/2} \rceil), 1 + \lceil 2^{j/2} \rceil] \right) \\
&\subset \delta \cdot \left([-2^{1+j}, 2^{1+j}] \times [-3 \cdot 2^{j/2}, 3 \cdot 2^{j/2}] \right),
\end{aligned}$$

since $1 + \lceil 2^{j/2} \rceil \leq 2 + 2^{j/2} \leq 3 \cdot 2^{j/2}$. Consequently, we get $A_{2^j}^{-1} [S_k^{-1} x + \delta n] \in \delta \cdot ([-2, 2] \times [-3, 3]) \subset [-3\delta, 3\delta]^2$. But by choice of C_0 from above, we have $\|\lambda^{-1}\|_{\ell^\infty \rightarrow \ell^\infty} \leq C_0$ and thus $\lambda^{-1} A_{2^j}^{-1} [S_k^{-1} x + \delta n] \in [-3\delta C_0, 3\delta C_0]^2$, so that $\omega(\lambda^{-1} A_{2^j}^{-1} [S_k^{-1} x + \delta n]) = 1$ for all $n \in n_0 + Z_j$. In combination with equation (C.3) and in view of $|Z_j| = (1 + 2 \cdot 2^j) (1 + 2 \cdot \lceil 2^{j/2} \rceil) \geq 2^{\frac{3}{2}j}$, we thus get

$$\sum_{m \in \mathbb{Z}^2} \omega(g^{-1}[x + \delta m]) \geq 2^{\frac{3}{2}j} \quad \forall x \in \mathbb{R}^2, (j, k) \in I \text{ and } g \in S_k A_{2^j} \Lambda.$$

Hence, equation (C.2) yields the following estimate:

$$\begin{aligned}
\sum_{m \in \mathbb{Z}^2} \langle T_F \gamma_{j,k,m}, \gamma_{j,k,m} \rangle &\geq 2^{\frac{3}{2}j} \cdot \int_H \mathbf{1}_{S_k A_{2^j} \Lambda}(g) \cdot \int_{\mathbb{R}^2} |\langle \gamma, \pi(x, g) \psi \rangle|^2 dx \frac{d\mu_H(g)}{|\det g|} \\
&= 2^{\frac{3}{2}j} \cdot \int_H \mathbf{1}_{S_k A_{2^j} \Lambda}(g) \cdot \|\langle \gamma, \pi(\bullet, g) \psi \rangle\|_{L^2}^2 \frac{d\mu_H(g)}{|\det g|}.
\end{aligned}$$

Now, with the modulation operator $[M_x f](\xi) = e^{2\pi i \langle x, \xi \rangle} \cdot f(\xi)$, Plancherel's theorem yields

$$\langle \gamma, \pi(x, g) \psi \rangle = \langle \gamma, L_g D_g \psi \rangle = \langle \hat{\gamma}, M_{-x} \mathcal{F}[D_g \psi] \rangle = \mathcal{F}^{-1} \left[\hat{\gamma} \cdot \overline{\mathcal{F}[D_g \psi]} \right](x),$$

where $\widehat{\gamma} \cdot \overline{\mathcal{F}[D_g \psi]} \in L^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$, since $\widehat{\gamma} \in L^2(\mathbb{R}^2)$ and $\overline{\mathcal{F}[D_g \psi]} = \overline{D_{g^{-T}} \widehat{\psi}} \in C_c^\infty(\mathbb{R}^2)$. Consequently, another application of Plancherel's theorem shows

$$\begin{aligned} \sum_{m \in \mathbb{Z}^2} \langle T_F \gamma_{j,k,m}, \gamma_{j,k,m} \rangle &\geq 2^{\frac{3}{2}j} \cdot \int_H \mathbf{1}_{S_k A_{2^j} \Lambda}(g) \cdot \|\langle \gamma, \pi(\bullet, g) \psi \rangle\|_{L^2}^2 \frac{d\mu_H(g)}{|\det g|} \\ &= 2^{\frac{3}{2}j} \cdot \int_H \mathbf{1}_{S_k A_{2^j} \Lambda}(g) \cdot \int_{\mathbb{R}^2} |\widehat{\gamma}(\xi)|^2 \cdot |(D_{g^{-T}} \widehat{\psi})(\xi)|^2 d\xi \frac{d\mu_H(g)}{|\det g|} \\ &= 2^{\frac{3}{2}j} \cdot \int_H \mathbf{1}_{S_k A_{2^j} \Lambda}(g) \cdot \int_{\mathbb{R}^2} |\widehat{\gamma}(\xi)|^2 \cdot \left| \widehat{\psi}(g^T \xi) \right|^2 d\xi d\mu_H(g) \\ &= \int_{\mathbb{R}^2} |\widehat{\gamma}(\xi)|^2 \int_H 2^{\frac{3}{2}j} \cdot \mathbf{1}_{S_k A_{2^j} \Lambda}(g) \cdot \left| \widehat{\psi}(g^T \xi) \right|^2 d\mu_H(g) d\xi \end{aligned}$$

for arbitrary $(j, k) \in I$. Now, set

$$G := \left\{ \xi \in \mathbb{R}^2 \mid |\xi_1| \in (0, 1) \text{ and } |\xi_2| \leq 8^{-1} \cdot |\xi_1|^{1/2} \right\}$$

$$\text{and } G_j := \left\{ \xi \in G \mid |\xi_1| \in [2^{-j-1}, 2^{-j}) \right\}, \quad \text{for } j \in \mathbb{N}_0$$

and observe $G = \bigsqcup_{j=0}^\infty G_j$. In view of equation (C.1), the preceding estimate implies

$$\infty > \sum_{(j,k) \in I} \int_G |\widehat{\gamma}(\xi)|^2 \int_H 2^{\frac{3}{2}j} \cdot \mathbf{1}_{S_k A_{2^j} \Lambda}(g) \cdot \left| \widehat{\psi}(g^T \xi) \right|^2 d\mu_H(g) d\xi. \quad (\text{C.4})$$

Now, we need the following auxiliary claim:

$$\forall j \in \mathbb{N}_0 \text{ and } k \in \mathbb{Z} \text{ with } |k| \leq \frac{1}{4} 2^{j/2} : \quad G_j \subset (S_k A_{2^j})^{-T} Q_0. \quad (\text{C.5})$$

To see that this is true, first note for $\xi \in \mathbb{R}^2$ that $\xi \in (S_k A_{2^j})^{-T} Q_0$ is equivalent to $(S_k A_{2^j})^T \xi \in Q_0$. By computing $(S_k A_{2^j})^T \xi$ explicitly, we thus get the following equivalence:

$$\begin{aligned} \xi \in (S_k A_{2^j})^{-T} Q_0 &\iff \begin{pmatrix} 2^j \xi_1 \\ 2^{j/2} (\xi_2 + k \xi_1) \end{pmatrix} \in Q_0 \\ &\iff 2^j |\xi_1| \in [2^{-1}, 1) \text{ and } \frac{2^{\frac{j}{2}} (\xi_2 + k \xi_1)}{2^j \xi_1} \in [-2^{-1}, 2^{-1}) \\ &\iff |\xi_1| \in [2^{-j-1}, 2^{-j}) \text{ and } \frac{\xi_2}{\xi_1} \in \left[-2^{\frac{j}{2}-1}, 2^{\frac{j}{2}-1} \right) - k. \end{aligned}$$

Hence, to prove the auxiliary claim (C.5), we only need to verify that this last condition is fulfilled for $\xi \in G_j$ and $|k| \leq \frac{1}{4} 2^{j/2}$. But $|\xi_1| \in [2^{-j-1}, 2^{-j})$ simply holds by definition of G_j . For the second condition, we note from the definition of G that

$$\left| \frac{\xi_2}{\xi_1} \right| \leq \frac{1}{8} \cdot |\xi_1|^{-1/2} \leq \frac{1}{8} \cdot (2^{-j-1})^{-1/2} = \frac{1}{8} \cdot 2^{\frac{j}{2} + \frac{1}{2}} < 2^{\frac{j}{2}-2}$$

and hence $\frac{\xi_2}{\xi_1} \in \left(-2^{\frac{j}{2}-2}, 2^{\frac{j}{2}-2} \right) \subset \left[-2^{\frac{j}{2}-1}, 2^{\frac{j}{2}-1} \right) - k$. Here, the last inclusion is indeed valid, since we have $|k| \leq \frac{1}{4} 2^{j/2}$ and thus

$$-2^{\frac{j}{2}-1} - k \leq -2^{\frac{j}{2}-1} + |k| \leq -\frac{1}{4} 2^{\frac{j}{2}} = -2^{\frac{j}{2}-2}, \quad \text{as well as} \quad 2^{\frac{j}{2}-1} - k \geq 2^{\frac{j}{2}-1} - |k| \geq \frac{1}{4} \cdot 2^{\frac{j}{2}} = 2^{\frac{j}{2}-2}.$$

Finally, note that $|k| \leq \frac{1}{4} 2^{j/2}$ in particular implies $|k| \leq \lceil 2^{j/2} \rceil$ and thus $(j, k) \in I$. Hence, a combination of equation (C.4) with the auxiliary claim (C.5) yields

$$\begin{aligned} \infty &> \sum_{(j,k) \in I} \int_G |\widehat{\gamma}(\xi)|^2 \int_H 2^{\frac{3}{2}j} \cdot \mathbf{1}_{S_k A_{2^j} \Lambda}(g) \cdot \left| \widehat{\psi}(g^T \xi) \right|^2 d\mu_H(g) d\xi \\ &\geq \sum_{j=0}^\infty \int_{G_j} 2^{\frac{3}{2}j} \cdot |\widehat{\gamma}(\xi)|^2 \sum_{k \in \mathbb{Z}, |k| \leq 2^{\frac{j}{2}-2}} \int_\Lambda \left| \widehat{\psi}((S_k A_{2^j} \lambda)^T \cdot \xi) \right|^2 d\mu_H(\lambda) d\xi \\ &\geq \mu_H(\Lambda) \cdot \sum_{j=0}^\infty \int_{G_j} 2^{\frac{3}{2}j} \cdot |\widehat{\gamma}(\xi)|^2 \cdot \left| \left\{ k \in \mathbb{Z} : |k| \leq 2^{\frac{j}{2}-2} \right\} \right| d\xi, \end{aligned}$$

where the last step used that each $\xi \in G_j$ satisfies $\xi \in (S_k A_{2^j})^{-T} Q_0$ and thus $(S_k A_{2^j} \lambda)^T \xi \in \Lambda^T Q_0 \subset Q$. Since $\widehat{\psi} \equiv 1$ on Q , this implies $\widehat{\psi} \left((S_k A_{2^j} \lambda)^T \xi \right) = 1$ for all $\lambda \in \Lambda$ and $\xi \in G_j$ and $k \in \mathbb{Z}$ with $|k| \leq \frac{1}{4} 2^{j/2}$.

Now, there are two cases: For $2^{\frac{j}{2}-2} \leq 1$, we have $\left| \left\{ k \in \mathbb{Z} : |k| \leq 2^{\frac{j}{2}-2} \right\} \right| \geq 1 \geq 2^{\frac{j}{2}-2}$. If otherwise $2^{\frac{j}{2}-2} > 1$, then $2^{\frac{j}{2}-2} \leq 1 + \lfloor 2^{\frac{j}{2}-2} \rfloor \leq 2 \cdot \lfloor 2^{\frac{j}{2}-2} \rfloor$, so that

$$\left| \left\{ k \in \mathbb{Z} : |k| \leq 2^{\frac{j}{2}-2} \right\} \right| \geq |\{-\lfloor 2^{\frac{j}{2}-2} \rfloor, \dots, \lfloor 2^{\frac{j}{2}-2} \rfloor\}| = 1 + 2 \cdot \lfloor 2^{\frac{j}{2}-2} \rfloor \geq 2^{\frac{j}{2}-2}$$

as well. Finally, for $\xi \in G_j$, we have $|\xi_1| \geq 2^{-j-1} = 2^{-(j+1)}$ and thus

$$2^{\frac{3}{2}j} \cdot 2^{\frac{j}{2}-2} = \frac{1}{16} \cdot 2^{2(j+1)} \geq \frac{1}{16} \cdot |\xi_1|^{-2}.$$

Putting everything together, we arrive at

$$\begin{aligned} \infty &> \mu_H(\Lambda) \cdot \sum_{j=0}^{\infty} \int_{G_j} 2^{\frac{3}{2}j} \cdot |\widehat{\gamma}(\xi)|^2 \cdot \left| \left\{ k \in \mathbb{Z} : |k| \leq 2^{\frac{j}{2}-2} \right\} \right| d\xi \\ &\geq \frac{\mu_H(\Lambda)}{16} \cdot \sum_{j=0}^{\infty} \int_{G_j} |\xi_1|^{-2} \cdot |\widehat{\gamma}(\xi)|^2 d\xi \\ &= \frac{\mu_H(\Lambda)}{16} \cdot \int_G |\xi_1|^{-2} \cdot |\widehat{\gamma}(\xi)|^2 d\xi, \end{aligned}$$

as desired. \square

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